GOEDEMENT-JACQUET INTEGRALS ON GL(n, C)

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Abstract. We explicitly compute the complex archimedean part of Godement-Jacquet zeta integral. We give a pair of a Schwartz-Bruhat function and a matrix coefficient which attains the local L-factor.

Introduction

As is well-known, Godement-Jacquet [2] established some fundamental properties of the standard L-function on the general linear group GL(n) via integral representation, as a generalization of Iwasawa-Tate theory. Our aim of this paper is to compute complex archimedean zeta integrals explicitly as in Tate’s thesis.

We review local archimedean theory of Godement-Jacquet [2]. Let \( F \) be an archimedean local field. Set \( A = \text{M}(n, \mathbb{F}) \) and \( G = \text{A}^x = \text{GL}(n, \mathbb{F}) \). Fix a maximal compact subgroup \( K \) of \( G \) by \( K = \text{O}(n) \) if \( F = \mathbb{R} \) and \( K = \text{U}(n) \) if \( F = \mathbb{C} \). We denote by \( | \cdot |_F \) the modulus of \( F \) defined by \( |x|_R = |x| \) and \( |x|_C = x\overline{x} = |x|^2 \) where \( | \cdot | \) is the ordinary absolute value.

Let \((\pi, V)\) be an irreducible admissible representation of \( G \), and \((\pi, \tilde{V})\) be its contragredient representation. We denote by \((\ , \ )\) the canonical pairing on \( V \). For \( v \in V \) and \( \tilde{v} \in \tilde{V} \), we define the function \( \beta_{v, \tilde{v}} \) on \( G \) by \( \beta_{v, \tilde{v}}(g) = (\pi(g)v, \tilde{v}) \) (\( g \in G \)). We call \( C(\pi) = \mathbb{C}\text{-span}\{\beta_{v, \tilde{v}} \mid v \in V, \tilde{v} \in \tilde{V}\} \) the space of matrix coefficients of \( \pi \). For \( \beta \in C(\pi) \), set \( \tilde{\beta}(g) = \beta(g^{-1}) \). Then we know that \( \tilde{\beta} \in C(\pi) \). We denote by \( S(A) \) be the space of Schwartz-Bruhat functions on \( A \). Let \( S_0(A) \) be the (dense) subspace of \( S(A) \) consisting of the functions of the form

\[
\begin{cases}
P(x_{ij}) \exp(-\pi \text{tr}(t^tx)) & \text{if } F = \mathbb{R}; \\
P(x_{ij}) \exp(-2\pi \text{tr}(t^tx)) & \text{if } F = \mathbb{C};
\end{cases}
\]

for \( x = (x_{ij}) \in A \), where \( P \) is a polynomial in \( x_{ij} \). Let \( \psi \) be the nontrivial additive character of \( F \) defined by

\[
\begin{cases}
\psi(x) = \exp(2\pi i x) & \text{if } F = \mathbb{R} \text{ and } x \in \mathbb{R}; \\
\psi(x) = \exp(2\pi i (x + \overline{x})) & \text{if } F = \mathbb{C} \text{ and } x \in \mathbb{C}.
\end{cases}
\]

For \( \Phi \in S(A) \), we define the Fourier transform \( \hat{\Phi} \) of \( \Phi \) with respect to \( \psi \) by

\[
\hat{\Phi}(x) = \int_A \Phi(y)\psi(\text{tr}(xy))d_{\psi}y,
\]

where \( d_{\psi}y \) is the self-dual Haar measure on \( A \). Note that \( \hat{\Phi} \in S_0(A) \) for \( \Phi \in S_0(A) \).

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For \( s \in \mathbb{C}, \beta \in C(\pi) \) and \( \Phi \in \mathcal{S}(A) \), the archimedean Godement-Jacquet zeta integral is defined by

\[
Z(s, \Phi, \beta) = \int_G \Phi(g)\beta(g) |\det g|_F^{s+(n-1)/2} \, dg.
\]

Here are fundamental properties of \( Z(s, \Phi, \beta) \) ([2], [3], [4]).

1. There exists \( s_0 \in \mathbb{R} \) such that \( Z(s, \Phi, \beta) \) converges absolutely for \( \text{Re}(s) > s_0 \).
2. There exists an \( L \)-factor such that for each \( \Phi \) and \( \beta \), the ratio \( Z(s, \Phi, \beta)/L(s, \pi) \) is a polynomial in \( s \). Moreover there exists a finite set \( (\Phi_i, \beta_i) \in \mathcal{S}_0(A) \times C(\pi) \), such that \( L(s, \pi) = \sum_i Z(s, \Phi_i, \beta_i) \).
3. There exists an \( \varepsilon \)-factor \( \varepsilon(s, \pi, \psi) \in \mathbb{C}^\times \) such that the local functional equation

\[
\frac{Z(1-s, \hat{\Phi}, \hat{\beta})}{L(1-s, \pi)} = \varepsilon(s, \pi, \psi) \frac{Z(s, \Phi, \beta)}{L(s, \pi)}
\]

holds.
4. The \( L \)- and \( \varepsilon \)-factors defined above coincide with the \( L \)- and \( \varepsilon \)-factors determined by the local Langlands correspondence.

In this paper, we shall give a pair \((\Phi, \beta)\) explicitly such that \( Z(s, \Phi, \beta) = L(s, \pi) \) when \( \pi \) is the irreducible principal series representation of \( GL(n, \mathbb{C}) \). We note that the explicit computation of \( Z(s, \Phi, \beta) \) for \( GL(2, \mathbb{R}) \) is explained in the textbook of Goldfeld and Huntley [3].

1. Representation theory of \( GL(n, \mathbb{C}) \)

1.1. Representation theory of \( K \). We first recall some basic facts on representation theory of \( K = U(n) \), that is, complex analytic finite dimensional representation of \( G = GL(n, \mathbb{C}) \). Let \( \Lambda = \{ \lambda = (\lambda_1, \dotsc, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \} \). For \( \lambda \in \Lambda \) we denote by \((\tau_\lambda, V_\lambda)\) the irreducible finite dimensional representation of \( G \) with the highest weight \( \lambda \). We recall the Gelfand-Tsetlin basis [1] of \( V_\lambda \). For \( \lambda \in \Lambda \), we denote by \( G(\lambda) \) the set of a pattern

\[
M = (m_{i,j})_{1 \leq i \leq j \leq n} = \begin{pmatrix}
m_{1,n} & m_{2,n} & \cdots & m_{n,n} \\
m_{1,n-1} & \cdots & m_{n-1,n-1} \\
\vdots & \ddots & \ddots \\
m_{1,1} & m_{2,2} & \cdots & m_{n,1}
\end{pmatrix}
\]

where the entries \( m_{i,j} \) satisfy the following:

\[
m_{i,n} = \lambda_i, \quad m_{i,j} - m_{i,j-1} \in \mathbb{Z}_{\geq 0}, \quad m_{i,j-1} - m_{i+1,j} \in \mathbb{Z}_{\geq 0}, \quad (1 \leq i \leq j \leq n).
\]

It is known that there exists a basis \( \{ v^\lambda_M \mid M \in G(\lambda) \} \) of \( V_\lambda \). If we denote the differential of \( \tau_\lambda \) by \( \tau_\lambda \) again, then

\[
\tau_\lambda(E_{k,k})v^\lambda_M = \left( \sum_{i=1}^k m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1} \right) v^\lambda_M;
\]

(1.1)

\[
\tau_\lambda(E_{k,k+1})v^\lambda_M = \sum_{j=1}^k \left| \prod_{i=1}^{k+1} (l_{i,k+1} - l_{j,k}) \prod_{i=1}^{k-1} (l_{i,k-1} - l_{j,k} - 1) \right|^{1/2} v^\lambda_{M+\delta_{j,k}};
\]

\[
\tau_\lambda(E_{k+1,k})v^\lambda_M = \sum_{j=1}^k \left| \prod_{i=1}^{k+1} (l_{i,k+1} - l_{j,k}) \prod_{i=1}^{k-1} (l_{i,k-1} - l_{j,k} + 1) \right|^{1/2} v^\lambda_{M-\delta_{j,k}};
\]
where $E_{i,j}$ is the matrix unit in $M(n, \mathbb{C})$ with 1 at $(i, j)$-th entry and 0 at other entries. Here $h_{i,j} = m_{i,j} - i$ and $M \pm \delta_{j,k}$ means $m_{j,k}$ is replaced by $m_{j,k} \pm 1$ in $M$.

**Remark 1.1.** The highest weight vector $v^\lambda_H$ of $\tau_\lambda$ is given by $H = (h_{i,j}) \in G(\lambda)$ with $h_{i,j} = \lambda_i$ for $1 \leq i \leq j \leq n$.

The $\mathbb{C}$-vector space $\{v^\lambda_M \mid M \in G(\lambda)\}$ admits the inner product $\langle \ , \ \rangle$ such that

\[
\langle v^\lambda_M, v^\lambda_N \rangle = \delta_{M,N} := \begin{cases} 1 & \text{if } M = N; \\ 0 & \text{if } M \neq N, \end{cases}
\]

and

\[
\langle \tau_\lambda(E_{i,j})v^\lambda_M, v^\lambda_N \rangle = \langle v^\lambda_M, \tau_\lambda(E_{j,i})v^\lambda_N \rangle, \quad (1 \leq i, j \leq n).
\]

Then we know that $\langle \ , \ \rangle$ is $K$-invariant inner product on $V_\lambda$.

\[
\tau_\lambda(k)u, \tau_\lambda(k)v = \langle u, v \rangle, \quad \forall u, v \in V_\lambda, \forall k \in K.
\]

For $M = (m_{i,j}) \in G(\lambda)$ and $l \in \mathbb{Z}$, we use the following notation:

- $\lambda + l = (\lambda_1 + l, \ldots, \lambda_n + l) \in \Lambda$
- $M + l := (m_{i,j} + l)_{1 \leq i \leq j \leq n} \in G(\lambda + l)$

**Lemma 1.2.** Let $M, N \in G(\lambda), c \in \mathbb{C}^\infty$ and $l \in \mathbb{Z}$.

1. We have $\langle \tau_\lambda(t^k) v^\lambda_M, v^\lambda_N \rangle = \langle \tau_\lambda(k) v^\lambda_N, v^\lambda_M \rangle$ for $k \in \mathbb{K}$.
2. We have $\forall g \in G$ \quad $\tau_\lambda(cg)v^\lambda_M = c^{\lambda_1 + \ldots + \lambda_n} \tau_\lambda(g)v^\lambda_M$.
3. We have $\det(k)^l \langle \tau_\lambda(k) v^\lambda_M, v^\lambda_N \rangle = \langle \tau_{\lambda+l}(k) v^{\lambda+l}_M, v^{\lambda+l}_N \rangle$ for $k \in \mathbb{K}$.

**Proof.** (1) is immediate from (1.2). (2) and (3) follow from (1.1).

For given $\lambda', \lambda'' \in \Lambda$, let us consider the decomposition $V_{\lambda'} \otimes V_{\lambda''} = \bigoplus_{\lambda \in \Lambda} m(\lambda', \lambda'') \lambda V_\lambda$.

Then, for each $v^\lambda_M \in V_\lambda$, we may denote by

\[
v^\lambda_M = \sum_{M' \in G(\lambda'), M'' \in G(\lambda'')} c^{M,r}_{M',M''} v^{\lambda'}_{M'} \otimes v^{\lambda''}_{M''},
\]

and call the number $c^{M,r}_{M',M''}$ Clebsch-Gordan coefficient. The following are obvious (cf. [5, 18.2]):

- For $M' \in G(\lambda')$ and $M'' \in G(\lambda'')$,

\[
v^\lambda_M \otimes v^{\lambda''}_{M''} = \sum_{\lambda \in \Lambda} \sum_{1 \leq M \in G(\lambda)} c^{M,r}_{M',M''} v^{\lambda'}_{M'} \otimes \overline{v^{\lambda''}_{M''}}.
\]

- Take $\lambda, \hat{\lambda} \in \Lambda$ with $m(\lambda', \lambda''); m(\lambda', \hat{\lambda}, \lambda''; \hat{\lambda}) \geq 1$. For $M \in G(\lambda)$, $\hat{M} \in G(\hat{\lambda})$, $1 \leq r \leq m(\lambda', \lambda''); 1 \leq \hat{r} \leq m(\lambda', \hat{\lambda}; \lambda'')$, we have

\[
\sum_{M' \in G(\lambda'), M'' \in G(\lambda'')} c^{M,r}_{M',M''} \overline{c^{M,\hat{r}}_{M',M''}} = \begin{cases} \delta_{M,\hat{M}} & \text{if } \lambda = \hat{\lambda} \text{ and } r = \hat{r}; \\ 0 & \text{otherwise}. \end{cases}
\]

Hereafter, for simplicity, we sometimes assume that the multiplicity $m(\lambda', \lambda''; \lambda)$ is one. In that case we denote a Clebsch-Gordan coefficient by $c^{M,r}_{M',M''}$ omitting the index $r$. Note that $m(\lambda', \lambda''; \lambda' + \lambda'') = 1$. 

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Lemma 1.3. Assume that \( m(\lambda', \lambda''; \lambda) = 1 \). Let \( M = (m_{i,j}) \in G(\lambda) \), \( M' = (m'_{i,j}) \in G(\lambda') \) and \( M'' = (m''_{i,j}) \in G(\lambda'') \). If \( c^M_{M', M''} \neq 0 \), then we have
\[
\sum_{i=1}^{j} (m'_{i,j} + m''_{i,j}) = \sum_{i=1}^{j} m_{i,j},
\]
for all \( 1 \leq j \leq n \).

Proof. Consider the action of \( E_{j,j} \) on \( v^\lambda_M = \sum_{M', M''} c^M_{M', M''} v^\lambda_{M'} \otimes v^{\lambda''}_{M''} \). \( \square \)

Corollary 1.4. For \( \lambda, \lambda', \lambda'' \in \Lambda \), assume that \( \lambda = \lambda' + \lambda'' \). Let \( v^\lambda_H \) be the highest weight vector in \( V_\lambda \) (Remark 1.1), \( M' = (m'_{i,j}) \in G(\lambda') \) and \( M'' = (m''_{i,j}) \in G(\lambda'') \). If \( c^H_{M', M''} \neq 0 \), then we have \( (M', M'') = (H', H'') \) where \( v^\lambda_{H'} \) and \( v^{\lambda''}_{H''} \) are the highest weight vectors in \( V_\lambda \) and \( V_{\lambda''} \), respectively. Furthermore we have \( \left| c^H_{H', H''} \right| = 1 \).

Proof. It is clear from Lemma 1.3 and (1.4). \( \square \)

Lemma 1.5. Assume that \( m(\lambda', \lambda''; \lambda) = 1 \). For given \( M, N \in G(\lambda) \), \( M' \in G(\lambda') \) and \( M'' \in G(\lambda'') \) we have
\[
\sum_{N' \in G(\lambda'), N'' \in G(\lambda'')} c_{N', N''}^N \int_K \langle \tau_{\lambda'}(k) v_{N'}^{\lambda'}, v_{M'}^{\lambda'} \rangle \langle \tau_{\lambda''}(k) v_{N''}^{\lambda''}, v_{M''}^{\lambda''} \rangle \langle \tau_{\lambda}(k) v_N^\lambda, v_M^\lambda \rangle dk = \frac{c^M_{M', M''}}{\dim V_\lambda}.
\]

Proof. In view of (1.3), we know that the integration over \( K \) can be written as
\[
\int_K \langle (\tau_{\lambda} \otimes \tau_{\lambda''})(k) (v_{N'}^{\lambda'}, v_{M'}^{\lambda'} \otimes v_{M''}^{\lambda''}), v_{M}^{\lambda} \rangle dk = \sum_{\mu \in \Lambda} \sum_{L, \ell \in G(\mu)} \sum_{1 \leq r, s \leq m(\lambda', \lambda''; \mu)} c_{N', N''}^{L, r} c_{M', M''}^{L, s} \int_K \langle \tau_{\mu}(k) v_{L}^{\mu}, v_{L}^{\mu} \rangle dk.
\]

Since
\[
\int_K \langle \tau_{\mu}(k) v_{L}^{\mu}, v_{L}^{\mu} \rangle dk = \begin{cases} (\dim V_\lambda)^{-1} \delta_{L, N} \delta_{L, M} & \text{if } \mu = \lambda; \\ 0 & \text{otherwise}, \end{cases}
\]
by Schur’s orthogonal relation, the assumption \( m(\lambda', \lambda''; \lambda) = 1 \) implies that (1.5) is equal to \( (\dim V_\lambda)^{-1} c_{N', N''}^{M, M''} \). Thus our assertion is immediate from (1.4). \( \square \)

1.2. Principal series representation. Let \( P = NM \) be the minimal parabolic subgroup of \( G \) where \( N = \{(x_{i,j}) \in G \mid x_{i,j} = 0 \ (1 \leq j < i \leq n), x_{i,i} = 1 \ (1 \leq i \leq n)\} \) and \( M = \{m = \text{diag}(m_1, \ldots, m_n) \in G \} \cong (\mathbb{C}^\times)^n \). For \( d = (d_1, \ldots, d_n) \in \mathbb{Z}^n \) and \( \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{C}^n \), define a character \( \chi_{\nu, d} \) of \( M \) by
\[
\chi_{\nu, d}(\text{diag}(m_1, \ldots, m_n)) = \prod_{i=1}^{n} \left( \frac{m_i}{|m_i|} \right)^{d_i} |m_i|^{\nu_i},
\]
and extend it the character on \( P \) by \( \chi_{\nu, d}(nm) = \chi_{\nu, d}(n,m) \) for \( (n, m) \in N \times M \). We define the function \( \rho \) on \( P \) by \( \rho(n \text{diag}(m_1, \ldots, m_n)) = \prod_{i=1}^{n} |m_i|^{m_1 + \nu_i} \) \( n \in N \). Set
\[
H_{\nu, d} = \{ f \in C^\infty(K) \mid f(mk) = \chi_{\nu, d}(m)f(k), \forall m \in M \cap K, \forall k \in K \},
\]
and define the principal series representation \( (\pi_{\nu, d}, H_{\nu, d}) \) of \( G \) by
\[
(\pi_{\nu, d}(g)f)(k) = \rho(p(kg)) \chi_{\nu, d}(p(kg))f(kg), \quad f \in H_{\nu, d}.
\]
where we denote by \( g = p(g)k(g) \) with \( p(g) \in P, k(g) \in K \). Note that the definition of \( \pi_{\nu,d} \) does not depend on (non-unique) \( PK \)-decomposition.

If \( \pi_{\nu,d} \) is irreducible, it is known that \( \pi_{\nu,d} \cong \pi_{w,\nu,w,d} := \pi_{(w_{\nu(1)}, \ldots, w_{\nu(n)}), (d_{w(1)}, \ldots, d_{w(n)})} \) for \( w \in \mathfrak{S}_n \). Then we may assume \( d_1 \geq d_2 \geq \cdots \geq d_n \), that is, \( d \in \Lambda \). The contragredient representation \( \pi_{\nu,d}^* \) of \( \pi_{\nu,d} \) is isomorphic to \( \pi_{-\nu,-d} ^* \) with \( d^* = (-d_n, \ldots, -d_1) \in \Lambda \) and \( \nu^* = (-\nu_n, \ldots, -\nu_1) \).

**Lemma 1.6.** For \( d \in \Lambda \) and \( M \in G(d) \), we define the function \( f_M(k) \) on \( K \) by

\[
 f_M(k) = \langle \tau_d(k)v_M^d, v_H^d \rangle.
\]

Here \( v_H^d \) is the highest weight vector in \( \tau_d \) (Remark 1.1). Then we have \( f_M \in H_{\nu,d} \).

**Proof.** Let \( m = \text{diag}(m_1, \ldots, m_n) \in M \cap K \), that is, \( |m_i|_C = 1 \). In view of (1.1), we know \( \tau_d(E_{k,k})v_H^d = (\sum_{i=1}^k d_i - \sum_{i=1}^{k-1} d_i)v_H^d = d_k v_H^d \). Then, by (1.2), we have

\[
 f_M(mk) = \langle \tau_d(k)v_M^d, \tau_d(m^{-1})v_H^d \rangle = \langle \tau_d(k)v_M^d, (\prod_{i=1}^n m_i^{-d_i})v_H^d \rangle
\]

\[
 = \langle \tau_d(k)v_M^d, \prod_{i=1}^n m_i^{-d_i} \rangle \langle \tau_d(k)v_M^d, v_H^d \rangle = \chi_{\nu,d}(m)f_M(k)
\]

as desired. \(\square\)

2. **Computation of archimedean zeta integrals**

When \( \pi \) is isomorphic to the irreducible principal series representation \( \pi_{\nu,d} \), let us compute \( Z(s, \Phi, \beta) \). Let \( f_1 \in \pi \) and \( f_2 \in \pi \). Then the integral

\[
 \beta_{f_1,f_2}(g) = \int_K f_1(kg)f_2(k) \, dk, \quad g \in G
\]

gives a matrix coefficient for \( \pi \). Using the Iwasawa decomposition \( G = NAK \) with

\[
 A = \{ a = \text{diag}(a_1, a_2, \ldots, a_n) \mid a_i > 0 \ (1 \leq i \leq n) \},
\]

we normalize a measure \( dg \) on \( G \) by

\[
 dg = dx \, d^\times a \, dk = \prod_{1 \leq i < j \leq n} d_{C|x_{ij}} \cdot \prod_{i=1}^n |a_i|^{-2a_i}_C a_i \cdot dk;
\]

for \( g = xak = (x_{ij}) \cdot \text{diag}(a_1, \ldots, a_n) \cdot k \). Here \( d_{C|x} = 2 dx_1 dx_2 (x = x_1 + \sqrt{-1}x_2) \) with the ordinary Lebesgue measure \( dx_1 \) on \( R \). Then we have

\[
 Z(s, \Phi, \beta_{f_1,f_2}) = \int_G \Phi(g) \int_K f_1(kg)f_2(k) |\det g|^s_{C}^{(n-1)/2} \, dk \, dg
 = \int_G \int_K \Phi(k^{-1}g)f_1(g)f_2(k) |\det g|^s_{C}^{(n-1)/2} \, dk \, dg
 = \int_{NA} \int_{K \times K} f_1(xak')f_2(k) \Phi(k^{-1}xak') |\det a|^s_{C}^{(n-1)/2} \, dk' \, dx \, d^\times a.
\]

If we denote by \( \Phi(g) = P_\Phi(g) \exp\{ -2\pi \text{tr}(\log g) \} \) \((g \in G)\) with a polynomial \( P_\Phi \) on \( G \), then we have

\[
 Z(s, \Phi, \beta_{f_1,f_2}) = \int_{C^{n-1/2}} \int_{(R_+)^n} I(a, x) \cdot \exp \left\{ -2\pi \left( \sum_{i=1}^n a_i^2 + \sum_{1 \leq i < j \leq n} x_{i,j}\bar{x}_{i,j} \right) \right\}
\]
\[
\times \prod_{i=1}^{n} a_i^{2(s+\nu_i)} \frac{2\pi a}{a_i} \prod_{1 \leq i < j \leq n} d_{C,x,i,j},
\]
where
\[
I(a, x) = \int_{K \times K} f_1(k') f_2(k) P_\Phi(k^{-1} x k') \, dk' dk.
\]

Therefore if we can choose \((f_1, f_2, \Phi)\) such that
\[
(2.1) \quad I(a, x) = \prod_{i=1}^{n} a_i^{\lfloor d_i \rfloor},
\]
the formulas
\[
\int_C \exp(-2\pi x \bar{x}) \, dC x = 1, \quad \int_{\mathbb{R}^+} \exp(-2\pi a^2) \frac{2\pi a}{a} = \Gamma_C(s)
\]
implies that
\[
Z(s, \Phi, \beta_{f_1, f_2}) = L(s, \pi) := \prod_{i=1}^{n} \Gamma_C(s + \nu_i + \frac{|d_i|}{2}).
\]

**Theorem 2.1.** For \(d = (d_1, \ldots, d_n) \in \Lambda\), we define \(d', d'' \in \Lambda\) by \(d' = (d_1, \ldots, d_t, 0, \ldots, 0)\) and \(d'' = (0, \ldots, 0, d_{t+1}, \ldots, d_n)\) where \(t\) is the largest integer such that \(d_i \geq 0\) and set \(\Delta = -\sum_{i=t+1}^{n} d_i \geq 0\). For \(M_1, M_2 \in G(d)\), let
\[
f_1(k) = (\dim V_d) \langle \tau_d(k) v_{M_1}, v_{H}^d \rangle, \quad f_2(k) = (\dim V_d) \langle \tau_d(k) v_{M_2}, v_{H}^d \rangle,
\]
and \(\Phi(g) = P_\Phi(g) \exp\{-2\pi tr(t g)\}\) with
\[
P_\Phi(g) = \sum_{P', Q' \in G(d')} C_{M_1}^{M_2} \langle \tau_{d'}(g) v_{P'}, v_{Q'}^d \rangle \langle \tau_{d''+\Delta}(\hat{g}) v_{P''+\Delta}^d, v_{Q''}^d \rangle
\]
where \(\hat{g} = (\det g)^{1/(n-1)} \hat{g}^{-1}\). Then we have
\[
Z(s, \Phi, \beta_{f_1, f_2}) = L(s, \pi).
\]

**Remark 2.2.** (The case of \(n = 2\)) Since the Clebsch-Gordan coefficients for \(U(2)\) are explicitly known, we can describe \(P_\Phi\) more precisely. Here is the tabular for \(d' = (d_1', d_2')\), \(d'' = (d_1'', d_2'')\) and \(\Delta\).

<table>
<thead>
<tr>
<th>(d_1 \geq d_2 \geq 0)</th>
<th>(d_1 \geq 0 \geq d_2)</th>
<th>(0 \geq d_1 \geq d_2)</th>
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<td>(d')</td>
<td>((d_1, d_2))</td>
<td>((d_1, 0))</td>
</tr>
<tr>
<td>(d'')</td>
<td>((0, 0))</td>
<td>((0, d_1))</td>
</tr>
<tr>
<td>(\Delta)</td>
<td>(0)</td>
<td>(-d_1)</td>
</tr>
</tbody>
</table>

Let us denote by \(v_p^\lambda := v_{(\lambda_1, \lambda_2)}^\lambda \) \((\lambda_2 \leq p \leq \lambda_1)\) and \(\gamma^\lambda_q = \sqrt{(\lambda_1 - \lambda_2)}\) \((0 \leq q \leq \lambda_1 - \lambda_2)\).

Then we can write \(f_1(k) = (d_1 - d_2 + 1) \langle \tau_d(k) v_{d_1'}, v_{d_1''}^d \rangle\), \(f_2(k) = (d_1 - d_2 + 1) \langle \tau_d(k) v_{d_1''}, v_{d_1'}^d \rangle\), \((d_2 \leq l, m \leq d_1)\) and
\[
P_\Phi(g) = \gamma^{d_1}_{d_2 - d_2} \gamma^{d_2}_{m - d_2} \sum_{P', Q', Q''} \gamma_{d' - d_2}^{d'} \gamma_{d'' - d_2}^{d''} \gamma_{d' - d_2}^{d' + \Delta} \gamma_{d'' - d_2}^{d'' + \Delta} \times \langle \tau_d'(g) v_{P'}, v_{Q'}^d \rangle \langle \tau_{d''+\Delta}(\hat{g}) v_{P''+\Delta}^d, v_{Q''}^d \rangle,
\]
where \(P', Q', P'', Q''\) run through such that \(P' + P'' = l, Q + Q' = m, 0 \leq P', Q' \leq d_1' - d_2'\) and \(0 \leq P'', Q'' \leq d_1'' - d_2''\).
Remark 2.3. If we take $M_1 = M_2 = H$, then Corollary 1.4 implies that

$$P_\Psi(g) = \langle \tau_{d'}(\bar{g}) v_{R'}^{d'}, v_{R'}^{d'} \rangle \langle \tau_{d''}(\bar{g}) v_{H'}^{d''+\Delta}, v_{H'}^{d''+\Delta} \rangle.$$ 

Proof of Theorem 2.1. We note that $f_1 \in \pi_{u,d}$ and $f_2 \in \tilde{\pi}_{u,d}$ by Lemma 1.6. We first compute $P_\Psi(k^{-1} x a k')$. Put $b = xa$. For $g = k^{-1} b k'$, we know $\tilde{g} = \det(\bar{k}k')^{1/(n-1)} \cdot \bar{t} k k'$. Then we have

$$\langle \tau_{d''}(\bar{g}) v_{P''}^{d''+\Delta}, v_{Q''}^{d''+\Delta} \rangle = (\det(\bar{k}k'))^\Delta \langle \tau_{d''}(\bar{k}k') v_{P''}^{d''+\Delta}, v_{Q''}^{d''+\Delta} \rangle$$

by Lemma 1.2 (2) and $|d'' + \Delta| = (n-1)\Delta$. By using

$$\tau_\lambda(g) v^\lambda_M = \sum_{R \in G(\lambda)} \langle \tau_\lambda(g) v^\lambda_M, v^\lambda_R \rangle v^\lambda_R,$$

we find that $P_\Psi(k^{-1} b k')$ becomes

$$\sum_{P''',Q'',R'',S'' \in G(d'')} c^M_{P'',P'} c^M_{Q'',Q'} \langle \tau_{d''}(\bar{k}) v^\delta_{S''}, v^\delta_{Q''} \rangle \langle \tau_{d''}(\bar{b}) v^\delta_{R''}, v^\delta_{S''} \rangle \langle \tau_{d''}(\bar{k}) v^\delta_{P''}, v^\delta_{P''} \rangle \langle \tau_{d''}(\bar{k}) v^\delta_{R''}, v^\delta_{P''} \rangle \langle \tau_{d''}(\bar{k}) v^\delta_{R''}, v^\delta_{Q''} \rangle$$

$$\times (\det(\bar{k}k'))^\Delta \langle \tau_{d''+\Delta}(\bar{k}) v^\delta_{S''+\Delta}, v^\delta_{Q''+\Delta} \rangle \langle \tau_{d''+\Delta}(\bar{b}) v^\delta_{R''+\Delta}, v^\delta_{S''+\Delta} \rangle \langle \tau_{d''+\Delta}(\bar{k}) v^\delta_{P''+\Delta}, v^\delta_{P''+\Delta} \rangle \langle \tau_{d''+\Delta}(\bar{k}) v^\delta_{R''+\Delta}, v^\delta_{R''+\Delta} \rangle.$$
and similarly
\[ \langle \tau^{d''+\Delta} (\bar{b}) v^{d''+\Delta}_{H''+\Delta}, v^{d''+\Delta}_{H''+\Delta} \rangle = \prod_{i=1}^{n-t} a_i^{-d_{t+i}}, \]
we arrive at (2.1) as desired. \( \square \)

REFERENCES


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