AN EKHOLM–SZŰCS-TYPE FORMULA FOR CODIMENSION ONE IMMERSIONS OF 3-MANIFOLDS UP TO BORDISM

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Abstract
We give a formula for the bordism class of an immersion of an oriented 3-manifold in 4-space. It expresses the class in terms of the topology of a null-cobordism of the 3-manifold and certain singularities (the number of umbilic points) of a generic map of this null-cobordism into 4-space which extends the immersion.

1. Introduction
Two immersions of oriented n-manifolds in $\mathbb{R}^{n+k}$ are said to be oriented bordant if there is an immersion of a compact oriented $(n+1)$-manifold with boundary in $\mathbb{R}^{n+k+1} \times [0,1]$ whose restriction to the boundary consists of the given immersions. The set $SI(n,k)$ of oriented bordism classes of immersions of n-manifolds in $\mathbb{R}^{n+k}$ is an abelian group under the operation of disjoint union of immersions (see [28]).

This paper deals with the group $SI(3,1)$ of oriented bordism classes of immersions of oriented 3-manifolds in $\mathbb{R}^4$. The group $SI(3,1)$ has been computed in [13] to be isomorphic to the cyclic group $\mathbb{Z}_{24}(=\mathbb{Z}/24\mathbb{Z})$ (see also [8, §4]). Our aim is to give a new formula for the isomorphism $SI(3,1) \approx \mathbb{Z}_{24}$, in terms of ‘singular Seifert surfaces’. Several applications are given in Section 4. We will consider smooth (that is, $C^\infty$-differentiable) manifolds and smooth immersions throughout.

2. Background

2.1. Singular Seifert surfaces
In knot theory, a Seifert surface for a given knot plays an important role. For instance, according to Hughes and Melvin [9], higher-dimensional knots are completely classified up to regular homotopy by the signatures of their Seifert surfaces. Ekholm and Szűcs [5] further generalised Hughes and Melvin’s work by introducing the notion of a singular Seifert surface, which has become an important tool in the study of geometric aspects of embeddings, immersions [5, 6, 10, 17, 18, 24] and other mappings [1] of manifolds (see also [2, 15, 23, 25] for other recent studies in this direction).

In our case, for an immersion $f : M^3 \hookrightarrow \mathbb{R}^4$ of an oriented 3-manifold in $\mathbb{R}^4$, its singular Seifert surface is a generic map $F : V^4 \to \mathbb{R}^4$ from a compact oriented 4-manifold $V^4$ with $\partial V^4 = M^3$, which has no singularity near the boundary and satisfies $F|_{\partial V^4} = f$. Such a singular Seifert surface always exists, since an immersion $f : M^3 \hookrightarrow \mathbb{R}^4$ of an oriented 3-manifold in $\mathbb{R}^4$ has trivial normal bundle.

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2.2. Generic maps from 4-manifolds and their umbilic points

Let $G: W^4 \to \mathbb{R}^4$ be a generic smooth map from a closed oriented 4-manifold into $\mathbb{R}^4$. Then it is known that the possible stable singularities of $G$ are:
- fold (or $A_1$-type) singularities,
- cusp (or $A_2$-type) singularities,
- swallowtail (or $A_3$-type) singularities,
- butterfly (or $A_4$-type) singularities, and
- umbilic (or $\Sigma^2,0$-type) singularities.

In particular, the set $\Sigma^2,0(G)$ of umbilic singularities of $G$ forms a 0-dimensional submanifold of $W^4$, and each umbilic singular point has an associated sign (see, for example, [20]). Denote by $\sharp\Sigma^2,0(G)$ the sum of the signs of the umbilic points of the map $G$. Then the Thom polynomial for umbilic singularities is calculated as
$$\sharp\Sigma^2,0(G) = 3\sigma(W^4),$$
where $\sigma(W^4)$ is the signature of $W^4$. For a concise review of the material here, see, for example, [16, §2].

2.3. The induced spin structure

Let $f: M^3 \to \mathbb{R}^4$ be an immersion of an oriented 3-manifold in $\mathbb{R}^4$. Then the standard trivialisation of $T\mathbb{R}^4$, via the bundle isomorphism
$$\varepsilon^1 \oplus TM^3 \cong f^*T\mathbb{R}^4,$$
induces a spin structure $s_f$ of $M^3$ (where $\varepsilon^1$ is the trivial one-dimensional vector bundle).

Recall also that the $\mu$-invariant (or Rohlin invariant) $\mu(M^3, s)$ of a spin 3-manifold $M^3$ (endowed with a spin structure $s$) is defined by
$$\mu(M^3, s) := \sigma(V^4) \pmod{16},$$
where $V^4$ is an arbitrary spin 4-manifold spin-bounded by $(M^3, s)$.

3. An Ekholm-Szűcs type formula for $\text{SI}(3, 1)$

The following is our main formula, which enables us to read off the oriented bordism class of a given immersion through its singular Seifert surface.

**Theorem 3.1.** The following is an isomorphism:
$$\text{SI}(3, 1) \to \mathbb{Z}_{24},$$
$$f \mapsto \frac{3(\sigma(V^4) - \mu(M^3, s_f)) - 2\sharp\Sigma^2,0(F)}{2} \pmod{24},$$
where $F: V^4 \to \mathbb{R}^4$ is a singular Seifert surface for $f$ and $\mu(M^3, s_f)$ denotes the $\mu$-invariant of $M^3$ with respect to the spin structure $s_f$ induced by $f$.

To prove Theorem 3.1, we will use the following notation and Lemma 3.2 below. Denote by $\text{Imm}[M^n, \mathbb{R}^{n+k}]$ the group of regular homotopy classes of immersions of $M^n$ in $\mathbb{R}^{n+k}$. Then the Smale invariant $\Omega$ (see [19]) and the inclusion $j: \mathbb{R}^4 \subset \mathbb{R}^5$ induce diagram (1), where each vertical arrow (the Smale invariant) is a group isomorphism and the generators are chosen according to [8, §2 and p. 180].
\[\begin{align*}
\text{Imm}[S^3, \mathbb{R}^4] & \xrightarrow{\imath} \text{Imm}[S^3, \mathbb{R}^5] \\
\Omega \mid_{\mathbb{Z} \oplus \mathbb{Z}} \mid_{n \rightarrow n + 2n} & \approx \Omega
\end{align*}\] (1)

**Lemma 3.2.** Let \( f : M^3 \hookrightarrow \mathbb{R}^4 \) be an immersion of a closed oriented 3-manifold \( M^3 \) in \( \mathbb{R}^4 \). Then the integer
\[a(f) := 3\sigma(V^4) - 2\Sigma^{2,0}(F),\]
where \( F : V^4 \rightarrow \mathbb{R}^4 \) is a singular Seifert surface for \( f \), does not depend on the choice of the singular Seifert surface, and is invariant up to regular homotopy of \( f \).

Furthermore, in the case \( M^3 = S^3 \), for an immersion \( f : S^3 \hookrightarrow \mathbb{R}^4 \) we have
\[a(f) = -2\Omega(j \circ f),\]
where \( \Omega(j \circ f) \) is the Smale invariant of the composition \( j \circ f : S^3 \hookrightarrow \mathbb{R}^5 \) of \( f \) with the inclusion \( j \).

**Proof of Lemma 3.2.** Let \( f_0 \) and \( f_1 : M^3 \hookrightarrow \mathbb{R}^4 \) be two immersions regularly homotopic to each other, and let \( h : M^3 \times [0, 1] \rightarrow \mathbb{R}^4 \times [0, 1] \) be a regular homotopy between them. Then, by using singular Seifert surfaces \( F_i : V_i^4 \rightarrow \mathbb{R}^4 \) for \( f_i \) \((i = 0, 1)\) and \( h \), we can construct a map
\[V_0^4 \cup V_0^4(M^3 \times [0, 1]) \cup V_0^4(-V_1^4) \rightarrow \mathbb{R}^4 \times [0, 1]\]
from the closed 4-manifold \( W^4 := V_0^4 \cup V_0^4(M^3 \times [0, 1]) \cup V_0^4(-V_1^4) \), obtained by gluing \( M^3 \times [0, 1] \) and \( V_1^4 \) \((i = 0, 1)\) along their common boundaries. By composing this map (suitably smoothed) with the projection \( \mathbb{R}^4 \times [0, 1] \rightarrow \mathbb{R}^4 \), we obtain a generic smooth map \( G : W^4 \rightarrow \mathbb{R}^4 \). The algebraic number \( 2\Sigma^{2,0}(G) \) of umbilic points of \( G \) is equal to \( 2\Sigma^{2,0}(F_0) - 2\Sigma^{2,0}(F_1) \), since the regular homotopy \( h : M^3 \times [0, 1] \rightarrow \mathbb{R}^4 \times [0, 1] \) is an immersion, and therefore its generic projection onto \( \mathbb{R}^4 \) has rank at least 3 everywhere.

By the Thom polynomial given in Section 2.2, we have \( 3\sigma(W^4) = 2\Sigma^{2,0}(G) \) and hence
\[3\sigma(V_0^4) - 2\Sigma^{2,0}(F_0) = 3\sigma(V_1^4) - 2\Sigma^{2,0}(F_1).\]
Thus, the first part of the claim is proved.

In the case where \( M^3 = S^3 \), the invariant \( a \) clearly gives rise to the homomorphism
\[a : \text{Imm}[S^3, \mathbb{R}^4] \rightarrow \mathbb{Z} \]
and we have to show that \( a \) coincides with the homomorphism
\[b : \text{Imm}[S^3, \mathbb{R}^4] \xrightarrow{j} \text{Imm}[S^3, \mathbb{R}^5] \xrightarrow{-2\Omega} \mathbb{Z}.
\]
Consider the subgroup \( E \subset \text{Imm}[S^3, \mathbb{R}^4] \) corresponding to
\[\{(m, n) \in \mathbb{Z} \oplus \mathbb{Z} \mid m + 2n \in 24\mathbb{Z}\}\]
via the Smale invariant \( \Omega : \text{Imm}[S^3, \mathbb{R}^4] \xrightarrow{\approx} \mathbb{Z} \oplus \mathbb{Z} \). Then [8, Theorem 3.1] shows that each class \((m, n)\) in \( E \) is represented by the boundary of an immersion \( F \) of the punctured manifold \( V^4 \setminus \text{Int} D^4 \) in \( \mathbb{R}^4 \) for a closed spin 4-manifold \( V^4 \) with Euler characteristic \( \chi(V^4) = m + 2 \) and signature \( \sigma(V^4) = -2(m + 2n)/3 \). Therefore, by [8, Theorem 3.1] and diagram (1) above, the two homomorphisms \( a \) and \( b \) coincide on the subgroup \( E \),
\[a(F|_{\partial(V^4 \setminus \text{Int} D^4)}) = b(F|_{\partial(V^4 \setminus \text{Int} D^4)}) = 3\sigma(V^4).
\]
Hence we have \( a = b \). This completes the proof. \(\square\)

**Remark 3.3.** We may want to extend the given immersion \( M^3 \hookrightarrow \mathbb{R}^4 \) by a generic map \( F \) of an oriented 4-manifold \( V^4 \) into \( \mathbb{R}^5_+ \) (the upper half-space of \( \mathbb{R}^5 \), bounded by \( \mathbb{R}^4 \)). In this case,
cusp points of the extension $F$ play a role, due to a result by Szücs [21, §2]. In fact, Ekholm and Szücs [5, Theorem 1.1; see also Remark 3.1] suggest a similar statement to Lemma 3.2 in terms of the signature $\sigma(V^4)$ and the number of cusp points of $F$. See also [1, §3.2].

**Remark 3.4.** The Smale–Hirsch theory [7] implies that for immersions $M^3 \hookrightarrow \mathbb{R}^4$, their regular homotopy classes bijectively correspond to homotopy classes of their induced stable framings (that is, trivialisations of $\varepsilon^3 \oplus TM^3$). In fact, the regular homotopy class of a given immersion $M^3 \hookrightarrow \mathbb{R}^4$ is characterised by the induced spin structure and the two integers, ‘the Hirzebruch defect’ and ‘the normal degree’ associated to the induced stable framing (see [11]). Actually, the invariant $a$ in Lemma 3.2 precisely equals the Hirzebruch defect of the induced stable framing.

**Proof of Theorem 3.1.** Let $f_0: M^3_0 \hookrightarrow \mathbb{R}^4$ and $f_1: M^3_1 \hookrightarrow \mathbb{R}^4$ be two immersions, oriented bordant to each other, and let $h: X^4 \hookrightarrow \mathbb{R}^4 \times [0, 1]$ be an oriented bordism between them. Then, by using singular Seifert surfaces $F_i: V^4_i \rightarrow \mathbb{R}^4$ for $f_i$ ($i = 0, 1$) and $h$, we obtain a generic map

$$G: V^4_0 \bigcup_{\mu} X^4 \bigcup_{\nu} (-V^4_1) \rightarrow \mathbb{R}^4$$

from the closed 4-manifold obtained by gluing $X^4$ and $V^4_i$ ($i = 0, 1$) along their common boundaries. This and the following steps are very similar to the proof of Lemma 3.2. That is, since $G$ has no umbilic points on $X^4$, which is originally immersed in $\mathbb{R}^4 \times [0, 1]$, we have

$$3(\sigma(V^4_0) - \mu(M^3_0, s_{f_0})) = \frac{1}{2} \Sigma^{2,0}(F_0) - \frac{1}{2} \Sigma^{2,0}(F_1).$$

The immersion $h$ induces, from the unique spin structure $\mathbb{R}^4 \times [0, 1]$, the spin structure on $X^4$, with which $X^4$ becomes a spin cobordism between $(M^3_0, s_{f_0})$ and $(M^3_1, s_{f_1})$. Therefore, we see that

$$\sigma(X^4) \equiv \mu(M^3_1, s_{f_1}) - \mu(M^3_0, s_{f_0}) \pmod{16}.$$

Thus, we have

$$3(\sigma(V^4_0) - \mu(M^3_0, s_{f_0})) - \frac{1}{2} \Sigma^{2,0}(F_0) \equiv 3(\sigma(V^4_1) - \mu(M^3_1, s_{f_1})) - \frac{1}{2} \Sigma^{2,0}(F_1) \pmod{48}.$$  

Hence, for a given immersion $f: M^3 \hookrightarrow \mathbb{R}^4$, by defining

$$c(f) := 3(\sigma(V^4) - \mu(M^3, s_f)) - \frac{1}{2} \Sigma^{2,0}(F)$$

using an arbitrary singular Seifert surface $F: V^4 \rightarrow \mathbb{R}^4$, we obtain the homomorphism

$$c: SI(3, 1) \rightarrow \mathbb{Z}_{48}.$$

Hughes [8, Theorem 2.3 and p. 180] has given an immersion $k: S^3 \hookrightarrow \mathbb{R}^4$ representing a generator of $SI(3, 1)$ (which is obtained by capping off a sphere eversion), and has shown that $\Omega(j \circ k) = 1$ (up to sign). Therefore, by Lemma 3.2, we have $c(k) \equiv a(k) \equiv -2\Omega(j \circ k) \equiv \pm 2 \pmod{48}$. This completes the proof.

Theorem 3.1 has the following direct corollary.

**Corollary 3.5.** For an immersion $f$ of an oriented 3-manifold in $\mathbb{R}^4$, the modulo 3 algebraic number $\frac{1}{2} \Sigma^{2,0}(F) \mod 3$ of the umbilic points of a singular Seifert surface $F$ is invariant up to oriented bordism of $f$.

**Remark 3.6.** Let $f: M^3 \hookrightarrow \mathbb{R}^4$ be an immersion. We can assume that $f$ is self-transversal after a small regular homotopy. Then, according to [22, Lemma 1.7(a)] (see also [4, §7.4]), we can define the epimorphism

$$\beta: SI(3, 1) \rightarrow \mathbb{Z}_8,$$
by the Brown invariant \([3]\) of the double-point surface of \(f\) with the induced \(\text{Pin}^-\)-structure. We call \(\beta(f)\) the Brown invariant of \(f\), which, together with the epimorphism

\[
\text{SI}(3, 1) \to \mathbb{Z}_3
\]

defined by the algebraic number (modulo 3) of umbilic points of a singular Seifert surface (see Corollary 3.5), also describes \(\text{SI}(3, 1)\).

The following is also an easy corollary of Theorem 3.1 (or of Lemma 3.2).

**Corollary 3.7.** If an immersion \(f: S^3 \hookrightarrow \mathbb{R}^4\) has a singular Seifert surface \(V^4 \to \mathbb{R}^4\) from a compact spin 4-manifold \(V^4\) with algebraically zero umbilic points, then \(f\) is regularly homotopic to the boundary of an immersion \(Y^4 \hookrightarrow \mathbb{R}^4\) of a compact spin 4-manifold \(Y^4\) with \(\sigma(Y^4) = \sigma(V^4)\).

**Proof.** If \(f\) has a singular Seifert surface from a compact spin 4-manifold \(V^4\) with algebraically zero umbilic points, then \(f\) is null-bordant, by Theorem 3.1. According to [8, p. 180], this implies that the regular homotopy class of \(f\) belongs to the kernel of

\[
\text{Imm}[S^3, \mathbb{R}^4] \longrightarrow \text{Imm}[S^3, \mathbb{R}^5] \xrightarrow{\Omega} \mathbb{Z} \xrightarrow{\text{proj.}} \mathbb{Z}_{24},
\]

which is exactly the subgroup \(E \subset \text{Imm}[S^3, \mathbb{R}^4]\) (see the proof of Lemma 3.2). Again by [8, Theorem 3.1], each class of \(E\) can be represented by the boundary of an immersion of a compact spin 4-manifold with suitable Euler characteristic and signature.

By using the singular Seifert surface \(V^4 \to \mathbb{R}^4\) and the immersion of \(Y^4 \hookrightarrow \mathbb{R}^4\), we can construct a generic smooth map \(V^4 \cup_{\partial} (-Y^4) \to \mathbb{R}^4\) with algebraically zero umbilic points. Therefore, we have \(\sigma(Y^4) = \sigma(V^4)\) (see Section 2.2).

\[
\text{Corollary 4.1.}\quad \text{Let } F: V^4 \to \mathbb{R}^4\text{ be a generic smooth map from a compact oriented 4-manifold } V^4\text{ with no singularity near the boundary } \partial V^4(\neq \emptyset).\text{ Then}
\]

\[
\sharp \Sigma_{2,0}^2(F) \equiv \sigma(V^4) - \mu_2(\partial V^4) \pmod{2}.
\]

**Proof.** We need only to look at the numerator of the formula in Theorem 3.1.

**Remark 4.2.** (i) Corollary 4.1, taken together with [27, Corollary 4.7], yields that if \(H_1(\partial V^4; \mathbb{Z}_2) = 0\), we have

\[
\sharp \Sigma_{2,0}^2(F) \equiv \sigma(V^4) - |H_1(\partial V^4; \mathbb{Z})| - 1 \pmod{2}
\]

\[
\equiv \dim_\mathbb{Z} H_2(V^4; \mathbb{Z}) - |H_1(\partial V^4; \mathbb{Z})| - 1 \pmod{2}.
\]
(ii) Similarly, together with [27, Corollary 4.8], Corollary 4.1 yields that if $H_1(\partial V^4; \mathbb{Q}) = 0$, we have

$$
\sharp \Sigma^2(F) \equiv \sigma(V^4) - \dim_{\mathbb{Z}_2} H_1(\partial V^4; \mathbb{Z}_2) \quad \text{(mod 2)}
$$

$$
\equiv \dim_{\mathbb{Z}} H_2(V^4; \mathbb{Z}) - \dim_{\mathbb{Z}_2} H_1(\partial V^4; \mathbb{Z}_2) \quad \text{(mod 2)}.
$$

Combining the formula in Theorem 3.1 with the Brown invariant $\beta$ in Remark 3.6, we have the following corollary.

**Corollary 4.3.** Let $F: V^4 \to \mathbb{R}^4$ be a generic smooth map from a compact oriented 4-manifold $V^4$ with no singularity near the boundary $\partial V^4(\neq \emptyset)$. Then

$$
\sharp \Sigma^{2,0}(F) \equiv 3(\sigma(V^4) - \mu(\partial V^4, s_{F|_{\partial V^4}})) - 2\beta(F|_{\partial V^4}) \quad \text{(mod 4)}.
$$

**Proof.** The conclusion follows from the fact that the isomorphism $\text{SI}(3,1) \to \mathbb{Z}_{24}$ in Theorem 3.1 and the epimorphism $\beta: \text{SI}(3,1) \to \mathbb{Z}_{8}$ coincide modulo 2.

The following theorem implies that not every immersion $S^3 \looparrowright \mathbb{R}^4$ can be lifted to an embedding in $\mathbb{R}^6$. Let $j: \mathbb{R}^4 \to \mathbb{R}^5$ be the inclusion.

**Theorem 4.4.** If an immersion $f: S^3 \looparrowright \mathbb{R}^4$ is regularly homotopic to the projection of an embedding $S^3 \hookrightarrow \mathbb{R}^6$, then the Brown invariant $\beta(f)$ of $f$ is even.

**Proof.** If $f$ is regularly homotopic to an immersion $f'$ which is the projection of an embedding in $\mathbb{R}^6$, then by [26, Theorem 3.2] the Smale invariant $\Omega(j \circ f')$ of the immersion $j \circ f': S^3 \looparrowright \mathbb{R}^5$ is even. Since $\Omega(j \circ f) = \Omega(j \circ f')$, by Lemma 3.2,

$$
3\sigma(V^4) - \sharp \Sigma^{2,0}(F) = -2\Omega(j \circ f) \equiv 0 \quad \text{(mod 4)},
$$

with respect to a singular Seifert surface $F: V^4 \to \mathbb{R}^4$ for $f$. This is equivalent to $\beta(f) \equiv 0 \mod 2$, by Corollary 4.3.

Theorem 4.4 implies that an immersion $S^3 \looparrowright \mathbb{R}^4$ with ‘complicated’ self-intersections cannot be lifted to an embedding in $\mathbb{R}^6$, while it can always be lifted to an embedding in $\mathbb{R}^7$. Concerning a possible converse of Theorem 4.4, we prove only the following weak proposition. Let $p_1: \mathbb{R}^6 \to \mathbb{R}^5$ and $p_2: \mathbb{R}^6 \to \mathbb{R}^4$ be the projections.

**Proposition 4.5.** An immersion $f: S^3 \looparrowright \mathbb{R}^4$ with even Brown invariant is bordant to the projection of an embedding $S^3 \hookrightarrow \mathbb{R}^6$.

**Proof.** By Corollary 4.3, $\beta(f) \equiv 0 \mod 2$ implies that $3\sigma(V^4) - \sharp \Sigma^{2,0}(F) \equiv 0 \mod 4$, where $F: V^4 \to \mathbb{R}^4$ is a singular Seifert surface for $f$. Hence, by Lemma 3.2, the Smale invariant $\Omega(j \circ f)$ is even. Then, according to [26, Corollary 3.4], $j \circ f$ is regularly homotopic to an immersion $S^3 \looparrowright \mathbb{R}^5$ which is the projection $p_1 \circ G$ of an embedding $G: S^3 \hookrightarrow \mathbb{R}^6$.

The proof of [14, Corollary: Multi-compression Theorem 4.5] implies that $G$ can be isotoped to an embedding $G'$ whose projection $p_2 \circ G': S^3 \looparrowright \mathbb{R}^4$ is an immersion, by an isotopy which covers a regular homotopy between $p_1 \circ G$ and $p_1 \circ G': S^3 \looparrowright \mathbb{R}^5$.

Thus, the immersions $j \circ f$ (which is regularly homotopic to $p_1 \circ G$) and $j \circ p_2 \circ G$ (which is regularly homotopic to $p_1 \circ G'$) are regularly homotopic; that is,

$$
\Omega(j \circ f) = \Omega(j \circ p_2 \circ G).
$$

Hence, by Lemma 3.2 and Theorem 3.1 (see also [8, p. 180]), $f$ is bordant to $p_2 \circ G$. 

\qed
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