Number of solutions for quartic simple Thue equations

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Abstract
Let $F(X, Y) = bX^4 - aX^3Y - 6bX^2Y^2 + aXY^3 + bY^4 \in \mathbb{Z}[X, Y]$.
We show that the number of solutions for the Thue equation $F(x, y) = \pm1$ is 0 or 4 except for a few already known cases. To obtain an upper bound for the size of solutions, we use Padé approximation method. To obtain a lower bound for the size of solutions, we construct a continued fraction with positive or negative rational partial quotients. This construction is carried out carefully by using special properties of the form $F$. Combining these lower and upper bounds, we obtain the result.

1 Introduction
Let $a, b$ be integers, and let

$$F(X, Y) = bX^4 - aX^3Y - 6bX^2Y^2 + aXY^3 + bY^4.$$ 

$F$ is called a quartic simple form. The aim of this paper is to show the following.

**Theorem 1.1** Let $a, b$ be non-zero integers. Then the number $N_F$ of solutions for the Thue equation

$$F(x, y) = \pm1 \quad (1)$$

is 0 or 4 except for the cases

$$b = 1, \quad a = \pm1, \pm4 \quad \text{with} \quad N_F = 8.$$
The case \( b = 1 \) was treated before. Lettl-Pethő [5] in 1995 solved the Thue equation (1) by Baker’s method for the case \( b = 1 \), and showed that the only solutions of (1) are \((\pm 1, 0)\) and \((0, \pm 1)\), except for \( b = 1 \), \( a = \pm 1, \pm 4 \).

Chen-Voutier [2] in 1997 obtained the same result as [5] by Padé approximation method. Lettl-Pethő-Voutier [6] in 1999 solved the Thue inequalities \(|F(x, y)| \leq 6a + 7\) for \( b = 1 \) by Padé approximation method realizing an idea of [3] for improvement of estimates. For the case of two parameters, we [11] in 2007 solved the Thue inequalities \(|F(x, y)| \leq 6a + 7b\) under the assumption \( a \geq 70b^{28/9} \) by Padé approximation method and by use of continued fractions with rational partial quotients. For the cubic simple Thue equations with two parameters \( F(3)(x, y) = bx^3 - ax^2y - (a + 3b)xy^2 - by^3 = 1 \), we [10] proved in 2007 that the number of solutions is 0 or 3, except for \( a = -1 \), \( b = 1 \) with 9 solutions and \( a = 0, 2 \), \( b = 1 \) with 6 solutions. To obtain this result, supposing there exists a solution we used Padé approximation method, and Okazaki’s gap principle [7] for general cubic Thue equations.

Outline of the proof of our theorem is as follows. We suppose that (1) has a solution \((x_0, y_0)\). Then,

(i) using \((x_0, y_0)\), by a standard Padé approximation method, we obtain an upper bound for the size of other solutions,

(ii) using \((x_0, y_0)\), we construct a continued fraction with positive or negative rational partial quotients, and obtain a lower bound for the size of other solutions, and

(iii) we show that the upper bound is smaller than the lower bound, which concludes that there is no other solution than \((x_0, y_0)\) (in a certain region).

The method for part (i) is now standard.

The main part of this article is part (ii). The construction of continued fraction with rational partial quotients is carried out as follows. Let \( \theta \) be the zero of \( F(x, 1) \) with \( \theta > 1 \). We choose \( x_0/y_0 \) for the first approximate rational number to \( \theta \). Note that in general this is not an integer. To choose the second approximate rational number \( x_1/y_1 \), we use Newton’s method. To choose the third approximate rational number \( x_2/y_2 \), we use, based on special properties of the form \( F(X, Y) \), the Taylor expansion of the function \( \sqrt{1 + x} \). At this step, we choose carefully the partial quotient so that its denominator becomes as small as possible (see §5.3). Then, we estimate the size of the denominators of the convergents, and the size of the common denominator of numerator and denominator of each convergent. We also estimate the size of the remainder terms of the continued fraction expansion. In order to obtain a lower bound for the size of other solutions, we use the following
basic estimate whose proof is easy.

Basic estimate. Let \( x'/y' \) and \( x/y \) be two different rational numbers such that \( |y'| \leq M \), and \( \frac{x'}{y'} - \frac{x}{y} < \frac{1}{MN} \). Then \( |y| > N \).

We apply this basic estimate to the other solutions \((x, y)\) than \((x_0, y_0)\), and the rational numbers \(x_0/y_0, x_1/y_1\) and \(x_2/y_2\) constructed by the continued fraction mentioned above, and we obtain a lower bound for \( y \).

2 Algebraic preliminaries

We give a characterization of the simple form \( F \).

Binary forms \( F_1, F_2 \in \mathbb{Z}[X, Y] \) are called equivalent over \( \mathbb{Q} \) if there exists \( g = (g_{ij}) \in GL(2, \mathbb{Z}) \) such that \( F_1^g := F_1(g_{11}X + g_{12}Y, g_{21}X + g_{22}Y) = rF_2(X, Y) \) with non-zero rational number \( r \).

**Proposition 2.1** Let \( \theta \) be an algebraic number of degree \( \geq 3 \). Suppose that the field \( \mathbb{Q}(\theta) \) is cyclic, and its Galois group is generated by a linear fractional transformation over \( \mathbb{Q} \). Then the degree of \( \theta \) is \( 3, 4 \), or \( 6 \), and the binary form corresponding to the defining polynomial for \( \theta \) is equivalent over \( \mathbb{Q} \) to \( F^{(3)}(X, Y) \), the above \( F(X, Y) \) or \( F^{(6)}(X, Y) = bX^6 - 2aX^5Y - (5a + 15b)X^4Y^2 - 20bX^3Y^3 + 5aX^2Y^4 + (2a + 6b)XY^5 + bY^6 \) respectively. These forms are called “simple forms”.

For the proof, see Ayad [1] for cubic case, and for general case see [6, Lemma 1] and [11, Remark 3.2].

For the parameters \( a, b \) of \( F \), we may suppose \( b > 0 \), and also \( a > 0 \) by symmetry of the shape of \( F \). Since the case \( b = 1 \) was already solved, we may suppose \( b > 1 \). If (1) has a solution, then \( gcd(a, b) = 1 \). Further, by considering congruence relation, we see that if (1) has a solution, then \( b \) is odd, and not divisible by 3. Therefore, in the following we assume

**Assumption 2.1** \( a > 0, \ gcd(a, b) = 1, \ gcd(b, 6) = 1, \) and \( b \geq 5 \).

If (1) has a solution \((x, y)\), then it implies 3 other solutions, as the following proposition shows.
Proposition 2.2 If \((x, y)\) is a solution of (1), then \((-x, -y), (-y, x), (y, -x)\) are also solutions of (1).

The proof is easy. By this proposition, the solutions of (1) are grouped four by four. Thus the number of solutions of (1) is a multiple of four, and our aim is to prove that (1) has at most one group of solutions.

The form \(F(X, Y)\) has quasi-automorphisms, that is,

Proposition 2.3 Let \(g = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\). Then \(F^g = F(X - Y, X + Y) = -4F(X, Y)\). Moreover, the linear fractional transformation \(z \mapsto \frac{z - 1}{z + 1}\) defined by \(g\) sends real intervals \((1, \infty)\) to \((0, 1)\), \((0, 1)\) to \((-1, 0)\), \((-1, 0)\) to \((-\infty, -1)\) and \((-\infty, -1)\) to \((1, \infty)\).

The proof is easy.

Let \((x, y)\) be a solution of (1). Then, \(x \not\equiv y \pmod{2}\), and \(F(x - y, x + y) = \mp 4\). Conversely, if \((x, y)\) is a solution of \(F(x, y) = \mp 4\), then, since \(b\) is odd by Assumption 2.1, we have \(x \equiv y \equiv 1 \pmod{2}\), and \((\frac{x-y}{2}, \frac{x+y}{2})\) gives a solution of (1). Moreover, if \(x/y\) belongs to one of the intervals in Proposition 2.3, then \((x - y)/(x + y)\) belongs to its image interval by the linear fractional transformation. Therefore, Theorem 1.1 is equivalent to the following, and we aim to prove it.

Theorem 2.1 Let \(m = 1, 4\). Then, under Assumption 2.1, the Thue equation

\[ F(x, y) = \pm m \quad (2) \]

has at most one solution with \(y > 0\) and \(x/y > 1\).

Using complex numbers \((i = \sqrt{-1})\), the binary form \(F\) can be written in a form having only two extreme terms as follows. Grace of this, Padé approximation method works well. This fact was already used in [2].

Proposition 2.4 The form \(F\) is written as

\[ F(X, Y) = \frac{1}{2}(\lambda(X + iY)^4 + \bar{\lambda}(X - iY)^4), \quad \lambda = b + \frac{a}{4}i. \]
The proof is given in [2, §3.1]. Alternatively, we can verify the formula by a simple calculation starting from the right-hand side, namely we have
\[
\left( b + \frac{a}{4}i \right) (X + iY)^4 = F(X, Y) + \frac{i}{4} \left( aX^4 + 16bX^3Y - 6aX^2Y^2 - 16bXY^3 + aY^4 \right).
\]

**Remark 2.1** The fact of the proposition is based on the invariant theory (see [4, p.72]). We explain this briefly. Let \( F \) be a quartic form, \( i \) be the invariant of degree 2, \( j \) be the invariant of degree 3, \( H \) be \( \frac{1}{12} \times \) Hessian, and \( J \) be the covariant of degree 3. Then we have the syzygy
\[
jF - H F^2 i + 4H^3 + J^2 = 0.
\]
For the simple form, we see \( i = -4C_0 \) with \( C_0 = -(b^2 + a^2/16) \), \( j = 0 \), and \( H = C_0X^4 + \cdots \). Hence we have \( HF^2i - 4H^3 = J^2 \). From this we see that \( H \) has a square factor. In fact, \( H = C_0H_1^2 \) with \( H_1 = X^2 + Y^2 \). We also see that \( J \) can be written as \( J = \frac{1}{2}C_0H_1J_1 \). Thus \( F^2 + \frac{1}{10}J_1^2 = -C_0H_1^4 \). From this we have \( F + \sqrt{-1}J_1 = \left( b + \frac{\sqrt{-1}a}{4}\right)(X + \sqrt{-1}Y)^4 \), which proves the proposition.

### 3 Analytic preliminaries

In the following we always assume Assumption 2.1 without mentioning it.

We put
\[
c = \frac{a}{b},
\]
so that
\[
F(x, y) = by^4 f(x/y).
\]
Let \( \theta, \theta', \theta'' \) and \( \theta''' \) be the zeros of \( f \) with \( \theta > 1 \), \( 0 < \theta' < 1 \), \( -1 < \theta'' < 0 \) and \( \theta''' < -1 \).

By technical reason, we proceed the proof of Theorem 2.1 dividing into 4 cases:
- **Case 1.** \( c \geq 5 \),
- **Case 2.** \( 3 \leq c < 5 \),
- **Case 3.** \( 7/6 \leq c < 3 \), and
- **Case 4.** \( 0 < c < 7/6 \).

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Lemma 3.1 The zeros of $f$ satisfy the following inequalities.

Case 1. $c + 4/c < \theta < c + 5/c$, $\theta' < 1 - 2/c + 3/c^2$, $\theta'' < -1/c + 5/c^3$, $\theta''' < -1 - 2/c$.

Case 2. $9c/11 + 19/11 < \theta < 9c/11 + 20/11$, $\theta' < 1 - 22/(9c + 31)$, $\theta'' < -11/(9c + 20)$, $\theta''' < -1 - 22/(9c + 9)$.

Case 3. $7c/10 + 21/10 < \theta < 7c/10 + 23/10$, $\theta' < 1 - 20/(7c + 33)$, $\theta'' < -10/(7c + 23)$, $\theta''' < -1 - 20/(7c + 13)$.

Case 4. $c/2 + 7/3 < \theta < c/2 + 5/2$, $\theta' < 1 - 4/(c + 7)$, $\theta'' < -2/(c + 5)$, $\theta''' < -1 - 4/(c + 3)$.

Proof. For Case 1, we verify $f(c + 4/c) < 0$ and $f(c + 5/c) > 0$, which implies the inequality for $\theta$ by the shape of the graph of $y = f(x)$. To obtain the inequalities for $\theta'$, $\theta''$, and $\theta'''$, we use the relations $\theta' = (\theta - 1)/(\theta + 1)$, $\theta'' = -1/\theta$ and $\theta''' = -(\theta + 1)/(\theta - 1)$. Similar for other cases too. See also [6, Lemma 9].

Lemma 3.2 Let $(x, y)$ be a solution of (2) with $y > 0$ and $x/y > 1$. Then $x/y$ belongs to the following interval:

Case 1. $c + 4/c < x/y < c + 5/c$.

Case 2. $9c/11 + 19/11 < x/y < 9c/11 + 20/11$.

Case 3. $7c/10 + 21/10 < x/y < 7c/10 + 23/10$.

Case 4. $c/2 + 7/3 < x/y < c/2 + 5/2$.

Proof. For Case 1, we verify $f(c + 4/c) < -1$ and $f(c + 5/c) > 1$, then from $4 \geq m = |F(x, y)| = by^4|f(x/y)| \geq 5y^4|f(x/y)| \geq 5|f(x/y)|$, we obtain the inequalities for $x/y$. Similar for other cases too.

Lemma 3.3 Let $(x, y)$ be a solution of (2) with $y > 0$ and $x/y > 1$. Then

$$\left|\frac{x}{y} - \theta\right| < \frac{m}{K(c)by^4},$$

where $K(c)$ is given as follows:

Case 1. $K(c) = c^3 + 16c - 3$.

Case 2. $K(c) = \frac{1}{11}(729c^3 + 4617c^2 + 14103c + 17046)$.

Case 3. $K(c) = \frac{1}{10^4}(343c^3 + 3087c^2 + 12061c + 18267)$.

Case 4. $K(c) = \frac{10}{8}c^3 + 2.19c^2 + 12.04c + 20.8$.

Proof. From $|F(x, y)| = by^4|(x/y - \theta)(x/y - \theta')(x/y - \theta'')(x/y - \theta')| = m$, Lemma 3.1 and Lemma 3.2, we obtain the above estimate.
4 Padé approximation

In order to obtain an upper bound for the size of solutions of (2), we use the standard Padé approximation method. Even though we know now a method of obtaining a sharper estimate than the classical method, that is, the improved method given by [6] following the idea of Chudnovsky [3], we use here the classical method since it is simpler and it already gives a sufficiently good estimate for our purpose.

We suppose that (2) has a solution, and we denote by \((x_0, y_0)\) the solution of (2) having the smallest \(y\) among the solutions \((x, y)\) with \(y > 0\) and \(x/y > 1\), and we put

\[ z_0 = \frac{x_0}{y_0}. \]

We fix this notation once for all.

The imaginary part of \(\left( b + \frac{x}{y}i \right) (x_0 + iy_0)^4\) given by Proposition 2.4 plays an important role in the following. So we put

\[ A = \frac{1}{m} (ax_0^4 + 16bx_0^3y_0 - 6ax_0^2y_0^2 - 16bx_0^3y_0^3 + ay_0^4). \]

As noticed in §2, if \((x, y)\) is a solution of (2) for \(m = 4\), then \(x\) and \(y\) are both odd. This implies that \(A\) is an integer.

We also put

\[ \gamma = \pm \frac{4}{Ai}, \tag{4} \]

where (and in the following also) we take the sign of \(\pm\) so that it coincides with (2). We have

\[ \lambda(x_0 + iy_0)^4 = F(x_0, y_0) + \frac{mA}{4} = \pm m \pm \frac{m}{\gamma} = \pm m \left( 1 + \frac{1}{\gamma} \right), \]

hence

\[ \overline{\lambda}(x_0 - iy_0)^4 = \frac{1 - 1/\gamma}{1 + 1/\gamma}. \]

On the other hand, since \(f(\theta) = 0\), we have \(\lambda(\theta + i)^4 + \overline{\lambda}(\theta - i)^4 = 0\), hence we have

\[ \frac{\overline{\lambda}}{\lambda} = \frac{(\theta + i)^4}{(\theta - i)^4}. \]

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Combining these two, we have
\[
\frac{(\theta + i)^4(x_0 - iy_0)^4}{(\theta - i)^4(x_0 + iy_0)^4} = 1 - \gamma
\]
\[
\frac{(\theta - i)(x_0 - iy_0)}{(\theta - i)(x_0 + iy_0)} = \frac{\sqrt{1 - \gamma}}{\sqrt{1 + \gamma}}.
\]
by taking the branch close to 1.

\textbf{Lemma 4.1} The value of \( A \) satisfies
\[
\frac{1}{m} A_1(c)by_0^4 < A < \frac{1}{m} A_2(c)by_0^4,
\]
where \( A_1(c) \) and \( A_2(c) \) are given as follows:

\textit{Case 1}. \( A_1(c) = c^5 + 26c^3 + 225c \), \( A_2(c) = c^5 + 30c^3 + 315c + 315 \),
\textit{Case 2}. \( A_1(c) = \frac{1}{114}(6561c^5 + 5540c^4 + 244944c^3 + 811224c^2 + 1406684c + 802560) \),
\( A_2(c) = \frac{1}{114}(6561c^5 + 58320c^4 + 263898c^3 + 882000c^2 + 1593377c + 982080) \),
\textit{Case 3}. \( A_1(c) = \frac{1}{104}(2401c^5 + 28812c^4 + 155134c^3 + 576828c^2 + 1309641c + 1145760) \),
\( A_2(c) = \frac{1}{104}(2401c^5 + 31556c^4 + 181006c^3 + 688436c^2 + 1637881c + 1578720) \),
\textit{Case 4}. \( A_1(c) = \frac{1}{16}c^5 + \frac{7}{5}c^4 + \frac{26}{7}c^3 + \frac{1064}{27}c^2 + \frac{9772}{81}c + \frac{4480}{27} \),
\( A_2(c) = \frac{1}{16}c^5 + \frac{5}{4}c^4 + \frac{79}{8}c^3 + \frac{185}{4}c^2 + \frac{2413}{16}c + 210 \).

\textbf{Proof.} Noting
\[
A = \frac{by_0^4}{m}(cz_0^4 + 16z_0^3 - 6cz_0^2 - 16z_0 + c),
\]
we put \( g(x) = cx^4 + 16x^3 - 6cx^2 - 16x + c \). For Case 1, we verify that \( g(x) \) is an increasing function in the interval \( (c + 4/c, c + 5/c) \), hence we have \( g(c + 4/c) < g(x_0/y_0) < g(c + 5/c) \), and estimating the both sides, we obtain the estimate for \( A \). Similar for other cases too.

In general, \( A \) is large, and we will consider the case where \( A \) is sufficiently large.

Using the smallest solution \( (x_0, y_0) \) of (2), we shall give an upper bound for the size of other solutions \( (x, y) \) by Padé approximation method. The
method we use here is based on Rickert’s integrals [8], and is very similar as used in [10, §4]. Therefore, we give only the result, omitting the proof, and just making remarks on the differences from [10].

In [10] we considered a Padé approximation to the function \( \sqrt[3]{(1 - x)/(1 + x)} \), which is now replaced by \( \sqrt[4]{(1 - x)/(1 + x)} \). The cubic root of unity \( \rho \) and the definition \( \gamma = 3\sqrt{-3}/A \) in [10] are replaced by \( i \) and \( \gamma = \pm 4/Ai \). Here we use an estimate \( b_{2n} := \left\lfloor \left( \frac{n+1}{2n} \right) \right\rfloor \leq \frac{5}{8} \left( \frac{1}{2} \right)^n \) instead of \( b_{2n} := \left\lfloor \left( \frac{n+1}{2n} \right) \right\rfloor \leq \frac{8}{9} \left( \frac{1}{2} \right)^n \) which is used in [10] implicitly, and in [9] explicitly. We use also the estimate \( A > 17 \) to obtain \( L > 1 \).

**Lemma 4.2** For \( n \geq 1 \), we have linear forms
\[
p_{in} + q_{in} \theta = l_{in}, \quad i = 0, 1
\]
satisfying the following conditions:

(i) the determinant \( \begin{vmatrix} p_{in} & q_{in} \\ p_{0n} & q_{0n} \end{vmatrix} \) is not zero,

(ii) \( |q_{in}| \leq \mu P^n \),

(iii) \( |l_{in}| \leq \mu L^{-n} \), and

(iv) \( p_{in} \) and \( q_{in} \) are integers,

where
\[
\mu = 1.67\sqrt[3]{1 + \sqrt{2}|\gamma||x_0 + iy_0|}, \quad P = 4(1 + \sqrt{2}|\gamma|)A, \\
l = \frac{5(\theta - i)(x_0 + iy_0)}{8(1 - |\gamma|^2)}, \quad L = \frac{(1 - |\gamma|^2)A}{16} > 1.
\]

From this lemma, we obtain the following lemma by a standard method. See for example [8, Lemma 2.1] and [9, Lemma 3].

**Lemma 4.3** For any integers \( p \) and \( q \) with \( q > 0 \), we have
\[
\left| \theta - \frac{p}{q} \right| > \frac{1}{C(a, b)q^\kappa},
\]

where
\[
\kappa = 1 + \frac{\log P}{\log L}
\]

and
\[
C(a, b) = 2\mu P(\max\{2l, 1\})^{(\log P)/(\log L)}.
\]
Lemma 4.4 If \( A \geq 131 \), then \( \kappa < 4 \), and if \( A \geq 37500 \), then \( \kappa < 2.536 \).

Proof. Since

\[
\kappa = 1 + \log \frac{P}{L} = 1 + \frac{\log(4(1 + 4\sqrt{2}/A)A)}{\log((1 - (4/A)^2)A/16)}
\]

is a decreasing function of \( A \), and its value for \( A = 131 \) is 3.9994, and its value for \( A = 37500 \) is 2.53599, we have the assertion.

From Lemma 4.3, we obtain an upper bound for the size of solutions of (2).

Lemma 4.5 Suppose \( \kappa < 4 \), and let \((x, y)\) be any solution of (2) with \( y > 0 \) and \( x/y > 1 \). Then

\[
y < \left( \frac{mC(a, b)}{K(c)b} \right)^{1/(4-\kappa)}.
\]

Proof. Combining Lemmas 3.3 and 4.3, we obtain

\[
\frac{1}{C(a, b)y^\kappa} < \left| \frac{x}{y} - \theta \right| < \frac{m}{K(c)y^4}.
\]

This implies the lemma.

Lemma 4.6 Suppose \( A \geq 37500 \), and let \((x, y)\) be any solution of (2) with \( y > 0 \) and \( x/y > 1 \). Then

\[
y < 7.43B(c)^{1.4}y_0^{1.74} \left( \frac{mA}{K(c)b} \right)^{0.69},
\]

where \( B(c) \) is given as follows:

Case 1. \( B(c) = c^2 + 12 \),
Case 2. \( B(c) = \frac{1}{10^7}(81c^2 + 360c + 521) \),
Case 3. \( B(c) = \frac{1}{10^9}(49c^2 + 322c + 629) \),
Case 4. \( B(c) = \frac{1}{4}(c^2 + 10c + 29) \).

Remark 4.1 The quantity \( B(c) \) is determined by the upper bound for \( |\theta - i| \) and \( |x_0/y_0 + i| \) given by Lemmas 3.1 and 3.2.
Remark 4.2 For $c > 1$, the lemma implies $y < \text{constant} \times c^{4.18} y_0^{4.5}$. This upper bound is not so large.

Proof. We apply Lemma 4.5 to the case $A \geq 37500$. By (6) and Lemma 4.3, we have
\[
C(a, b) = 2\mu P(\max\{2l, 1\})^{(\log P)/(\log L)}
\]
\[
= 2 \cdot 1.67 \sqrt[3]{1 + \sqrt{2}/|\gamma||x_0 + iy_0|} \cdot 4(1 + \sqrt{2}/|\gamma|) A \left( \frac{2 \cdot 5}{8(1 - |\gamma|^2)} \right)^{\log P/\log L}.
\]
By Lemma 4.4, we have $\kappa < 2.536$, hence $(\log P)/(\log L) < 1.536$. From Lemmas 3.1 and 3.2, we have
\[
|\theta - i| < B(c)^{1/2}, \quad |x_0 + iy_0| < B(c)^{1/2} y_0,
\]
with $B(c)$ given in the lemma. We evaluate $|\gamma|$ by $|\gamma| = 4/A \leq 4/37500$. Putting all these values, we obtain
\[
C(a, b) < 18.83 B(c)^{2.036} y_0^{2.536} A,
\]
and from this and Lemma 4.5, we obtain the lemma.

5 Lower bound for solutions

Here we shall obtain a sufficiently large lower bound for the size of other solutions than $(x_0, y_0)$.

5.1 Elementary lower bound

We begin with giving a lower bound which is obtained elementarily.

**Lemma 5.1** Let $(x, y)$ be a solution of (2) with $y > 0$, $x/y > 1$, and $x/y \neq x_0/y_0$. Then
\[
y > K(c)b y_0^3/2m.
\]
Proof. From Lemma 3.3, we have
\[
\frac{x_0}{y_0} - \theta < \frac{m}{K(c)b y_0^4}, \quad \frac{x}{y} - \theta < \frac{m}{K(c)b y^4}.
\]
Since $x/y \neq x_0/y_0$ and $y \geq y_0$, we obtain

$$\frac{1}{y_0 y} \leq \left| \frac{x_0}{y_0} - \frac{x}{y} \right| \leq \left| \frac{x_0}{y_0} - \theta \right| + \left\| \theta - \frac{x}{y} \right\| < \frac{2m}{K(c)by_0^2},$$

which implies the lemma by the basic estimate.

This lower bound is not sufficient to obtain a contradiction with the upper bound of Lemma 4.6 (see Remark 4.2).

### 5.2 Continued fraction with rational partial quotients

In order to obtain a lower bound for solutions of (2), we first give a method in general form. We consider continued fractions with rational partial quotients. We allow for these partial quotients to be positive or negative rational numbers.

Let $\xi$ be a real number. For $\xi$ we choose a rational number $k_0$ satisfying

$$k_0 - 1 < \xi < k_0 + 1,$$

and define $\xi_1$ by

$$\xi = k_0 + \frac{1}{\xi_1}.$$

We have $|\xi_1| > 1$. Then we choose a rational number $k_1$ satisfying

$$k_1 - 1 < \xi_1 < k_1 + 1.$$

Since $|\xi_1| > 1$, we have $k_1 \neq 0$. We define $\xi_2$ by

$$\xi_1 = k_1 + \frac{1}{\xi_2}.$$

We continue this process. For $i \geq 1$ we choose a rational number $k_i$ satisfying

$$k_i - 1 < \xi_i < k_i + 1,$$

and define $\xi_{i+1}$ by

$$\xi_i = k_i + \frac{1}{\xi_{i+1}}.$$
Since $|\xi_i| > 1$, we have $k_i \neq 0$ for $i \geq 1$. Then we have

$$\xi = [k_0, k_1, k_2, \ldots, k_n, \xi_{n+1}] = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \cdots + \frac{1}{k_n + \xi_{n+1}}}}.$$ 

We call this or

$$\xi = [k_0, k_1, k_2, \cdots, k_n, \xi_{n+1}]$$

a continued fraction expansion with rational partial quotients for $\xi$. Note that this expansion is not unique. If $\xi$ is a rational number, we may stop the process at a certain step by putting for example $k_{n+1} = \xi_{n+1}$. The convergents $p_n/q_n$ are defined by

\[
\begin{align*}
    p_0 &= 1, \quad p_1 = k_0, \quad p_{n+1} = k_n p_n + p_{n-1} \quad (n \geq 1), \\
    q_0 &= 0, \quad q_1 = 1, \quad q_{n+1} = k_n q_n + q_{n-1} \quad (n \geq 1).
\end{align*}
\]

By this definition, we have for $n \geq 1$

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^n,$$

$$\xi = \frac{p_n \xi_n + p_{n-1}}{q_n \xi_n + q_{n-1}}$$

and

$$\left| \frac{\xi - p_n}{q_n} \right| = \frac{1}{|q_n(q_n \xi_n + q_{n-1})|}.$$ 

Note that the series $p_n/q_n$ does not necessarily converge to $\xi$. For $n \geq 0$, we take a positive rational number $d_n$ such that $d_n p_n, d_n q_n \in \mathbb{Z}$. Here we do not necessarily assume that $d_n$ is the smallest one with this property. Of course, a smaller $d_n$ is better. In case where $d_n$ is a natural number, we may call it a common denominator of $p_n$ and $q_n$.

**Lemma 5.2** Let $\xi$ be a real number, let $p_n/q_n$ be the $n$-th convergent defined by a continued fraction expansion for $\xi$ with (positive or negative) rational partial quotients, and let $\xi_n$ be the $n$-th remainder term. Let $d_n$ be a positive rational number such that $d_n p_n, d_n q_n \in \mathbb{Z}$. Let $x, y$ be integers with $y \neq 0$ and $\gcd(x, y) = 1$. Suppose

$$\left| \frac{x}{y} - \xi \right| < \frac{m}{K|y|^d},$$

for some $m > 0$, $K > 0$, $d > 0$. Suppose further

$$\frac{m}{K|y|^d} < \frac{c_n}{|q_n(q_n \xi_n + q_{n-1})|}.$$
for some $0 < c_n \leq 1$. Then

$$|y| > \frac{|q_n\xi_n + q_{n-1}|}{(1 + c_n)d_n}. \quad (9)$$

**Remark 5.1** Two constants $m$ and $K$ may be put together to one constant, but viewing the form of Lemma 3.3, we leave them as above. Two inequalities (7) and (8) also may be put together, omitting $m/K|y|^d$.

**Proof.** From general properties of continued fraction expansion, we have

$$\left| \frac{p_n}{q_n} - \xi \right| = \frac{1}{|q_n(q_n\xi_n + q_{n-1})|}. \quad (10)$$

First we show $\frac{x}{y} \neq \frac{p_n}{q_n}$ as rational numbers. To show this, suppose $x/y = p_n/q_n$. Then, from (7),(8) and (10), we would have

$$\frac{c_n}{|q_n(q_n\xi_n + q_{n-1})|} > \frac{m}{Ky^d} > \left| \frac{x}{y} - \xi \right| = \left| \frac{p_n}{q_n} - \xi \right| = \frac{1}{|q_n(q_n\xi_n + q_{n-1})|},$$

a contradiction.

Next, since $x/y \neq p_n/q_n$, we have

$$\left| \frac{x}{y} - \frac{p_n}{q_n} \right| = \left| \frac{xq_n - yp_n}{yq_nd_n} \right| \geq \frac{1}{|yq_n|d_n},$$

because the numerator is a non-zero integer. On the other hand, from (7),(8) and (10), we have

$$\left| \frac{x}{y} - \frac{p_n}{q_n} \right| < \left| \frac{x}{y} - \xi \right| + \left| \xi - \frac{p_n}{q_n} \right| < \frac{c_n + 1}{|q_n(q_n\xi_n + q_{n-1})|}.$$

Hence we have

$$\frac{1}{|yq_n|d_n} < \frac{c_n + 1}{|q_n(q_n\xi_n + q_{n-1})|},$$

from which we obtain (9).
5.3 Continued fraction for $\theta$

In order to obtain a larger lower bound, we construct a continued fraction with rational partial quotients for $\theta$. For this we first use Newton’s method. Viewing that $z_0 = x_0/y_0$ is close to $\theta$, we draw the tangential line to the curve $y = f(x)$ at the point $(z_0, f(z_0))$, and put $x_1/y_1$ to be the $x$-coordinate of the intersection point of this tangential line and $x$-axis: $x_1/y_1 = z_0 - f(z_0)/f'(z_0)$. From (2) and (3), we obtain

$$\frac{x_1}{y_1} = z_0 + \frac{1}{-f'(z_0)/f(z_0)} = z_0 + \frac{1}{f(z_0)/f'(z_0)y_0/m} =: k_0 + \frac{1}{k_1}.$$

For later use, we show the following.

**Lemma 5.3** We have

$$\frac{f'(z_0)}{f(z_0)} = \pm A + 4z_0 \frac{z_0^2 + 1}{z_0^2 + 1}.$$

**Proof.** First we show

$$F_X(X, Y)(X^2+Y^2) = 4F(X, Y)X+(aX^4+16bX^3Y-6aX^2Y^2-16bXY^3+aY^4)Y.$$

To show this, we differentiate by $X$ the formula in Proposition 2.4, multiply the result by $(X+iY)(X-iY)$, and use the formula in the proof of Proposition 2.4, then we obtain the above formula.

We put $X = x_0$, $Y = y_0$ into the above formula. Then

$$\frac{F_X(x_0, y_0)}{F(x_0, y_0)} = \frac{4x_0}{x_0^2 + y_0^2} + \frac{(ax_0^4 + 16bx_0^3y_0 - 6ax_0^2y_0^2 - 16bx_0y_0^3 + ay_0^4)y_0}{F(x_0, y_0)(x_0^2 + y_0^2)}$$

$$= \frac{4x_0}{x_0^2 + y_0^2} + \frac{(ax_0^4 + 16bx_0^3y_0 - 6ax_0^2y_0^2 - 16bx_0y_0^3 + ay_0^4)y_0}{\pm m(x_0^2 + y_0^2)},$$

and by definition of $A$, this implies the desired formula.

In order to obtain further terms of continued fraction expansion for $\theta$, we use the expression (5) for $\theta$ and Taylor expansion of $\sqrt{1+x}$. Continued fraction expansion with rational partial quotients is not unique. However, if its partial quotients have smaller denominators, it is better. So, at each step of expansion, we choose carefully partial quotient as better as possible. We shall show the following.
Lemma 5.4 For \( \theta \) we have a continued fraction expansion with rational partial quotients as follows:

\[
\theta = \frac{x_0}{y_0} + \frac{1}{-(\pm A y_0 + 4x_0)y_0 + \frac{1}{x_0^2 + y_0^2 + \xi_2}}
\]

\[
= k_0 + \frac{1}{k_1 + \frac{1}{\xi_2}} = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \frac{1}{\xi_3}}},
\]

with

\[
k_0 = \frac{x_0}{y_0}, \quad k_1 = -\frac{f'(z_0)}{f(z_0)} = -\frac{(\pm A y_0 + 4x_0)y_0}{x_0^2 + y_0^2}, \quad k_2 = \frac{x_0^2 + y_0^2}{3x_0y_0},
\]

\[
\xi_2 = \frac{x_0^2 + y_0^2}{3x_0y_0} \left(1 + \frac{\xi_2}{A}\right), \quad \xi_3 = \pm \frac{9x_0^2 A}{5(x_0^2 + y_0^2)} \left(1 + \frac{2\xi_3}{A}\right).
\]

Further, if \( A \geq 37500 \), then \(|\varepsilon_2|, |\varepsilon_3| < 1\).

Proof. From (5) we have

\[
\theta = z_0 + \frac{-(1 + z_0^2)}{z_0 + \frac{\xi_0 + \frac{\xi_0^2}{2}}{\varepsilon_2 + \frac{\xi_2}{A}}}.
\]

If \( z \) is a purely imaginary complex number, then the remainder term of Taylor’s expansion

\[
\sqrt{1 + z} = 1 + \left(\frac{1}{4} \right) z + \cdots + \left(\frac{1}{n - 1}\right) z^{n-1} + \left(\frac{1}{n}\right) \varepsilon z^n \quad (\varepsilon \in C)
\]

satisfies \(|\varepsilon| < 1\), since \( \max_{0 < t \leq 1} |(1 + tz)^{1/4 - n}| < 1 \) for \( n \geq 1 \). Using this, (4) and \( A \geq 37500 \), we first obtain

\[
\theta = z_0 + \frac{-(1 + z_0^2)}{\pm A + z_0 \pm \frac{5(1 + \frac{\xi_0^2}{2}\frac{\xi_0}{\varepsilon_2})}{A(1 + \frac{\xi_0^2}{2}\frac{\xi_0}{\varepsilon_2})}},
\]

with \(|\varepsilon_1'|, |\varepsilon_0'| < 1\). By lemma 5.3 we see that \( \pm A + 4z_0 \) has smaller denominator than \( \pm A + 3z_0 \), hence we change \( \pm A + z_0 \) to \( (\pm A + 4z_0) - 3z_0 \). Thus we have

\[
\theta = z_0 + \frac{-(1 + z_0^2)}{(\pm A + 4z_0) - 3z_0 \pm \frac{5(1 + \frac{\xi_0^2}{2}\frac{\xi_0}{\varepsilon_2})}{A(1 + \frac{\xi_0^2}{2}\frac{\xi_0}{\varepsilon_2})}}.
\]
We can rewrite this in the form of the lemma, and using the estimates $\frac{1+t}{1-s} < 1 + 2(s + t)$ and $\frac{1-t}{1+s} > 1 - (s + t)$ which hold for $1/2 > s > 0$, $t > 0$, we obtain the estimate for $|\varepsilon_2|, |\varepsilon_3|$ also.

Now we define $p_i, q_i$ as usual and use Lemma 5.3, then

$p_0 = 1, \quad p_1 = k_0 = \frac{x_0}{y_0},
q_0 = 0, \quad q_1 = 1,$

$p_2 = k_1p_1+p_0 = -\frac{f'(z_0)}{f(z_0)} \cdot \frac{x_0}{y_0} + 1 = \mp \frac{bf'(x_0/y_0)x_0y_0^2}{m} + 1 = -\frac{(\pm Ay_0 + 4x_0)x_0}{x_0^2 + y_0^2} + 1,$

$q_2 = k_1q_1 + q_0 = -\frac{f'(z_0)}{f(z_0)} = \mp \frac{bf'(x_0/y_0)y_0^3}{m} = -\frac{(\pm Ay_0 + 4x_0)y_0}{x_0^2 + y_0^2},$

$p_3 = k_2p_2 + p_1 = \mp A x_0 + y_0 \quad \frac{3x_0}{3x_0},
q_3 = k_2q_2 + q_1 = \mp A y_0 - x_0 \quad \frac{3x_0}{3x_0}.$

Then

$$\frac{x_1}{y_1} = \frac{p_2}{q_2}.$$ We put

$$\frac{x_2}{y_2} = \frac{p_3}{q_3}.$$ Note that $x_n$ and $y_n$ $(n = 0, 1, 2)$ are positive integers and $\gcd(x_n, y_n) = 1$, while in general, $p_n$ and $q_n$ are rational numbers for $n \geq 1$. For $n \geq 1$, we take an integer $d_n$ such that $d_np_n, d_nq_n \in N$.

**Lemma 5.5** We can take

$$d_1 = y_0, \quad d_2 = 2, \quad d_3 = 3x_0.$$  

**Proof.** (i) For $d_1$, it is clear.

(ii) For $d_2$, in case $m = 1$ we see $p_2, q_2 \in Z$. In case $m = 4$, we first see that $by_0^3f'(x_0/y_0) = 4bx_0^3 - 3ax_0^2y_0 - 12bx_0y_0^2 + ay_0^3$ is even since $x_0$ and $y_0$ are odd. Hence $q_2 = \mp by_0^3f'(x_0/y_0)y_0/m \in \frac{1}{2}Z$. Also, $p_2 \in \frac{1}{2}Z$. Thus $2p_2, 2q_2 \in Z$.

(iii) For $d_3$, from the expression of $p_3, q_3$, we see $3x_0p_3, 3x_0q_3 \in Z$ since $A \in Z$.  

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5.4 Comparison with $x_1/y_1$

In order to obtain a lower bound for solutions $(x, y)$ of (2), we first compare them with $x_1/y_1$ using Lemmas 5.1 and 5.2. For it, we prepare the following estimate.

**Lemma 5.6** Let $(x, y)$ be a solution of (2) with $y > 0$, $x/y > 1$, and $x/y \neq x_0/y_0$. Suppose $A \geq 37500$. Then

$$\frac{m}{K(c)by^4} < \frac{c_2}{|q_2(q_2\xi_2 + q_1)|},$$

where $c_2$ is given as follows:

- Case 1. $c_2 = 0.0000033$,
- Case 2. $c_2 = 0.000021$,
- Case 3. $c_2 = 0.00033$,
- Case 4. $c_2 = 0.0021$.

**Proof.** We first show that, if $A \geq 37500$, then

$$z_0/A < 0.0008. \quad (11)$$

When $c \geq 5$, from Lemma 3.2, Lemma 4.1, and $by_0^4/m > 1$, we have $z_0/A < z_0/(by_0^4A_1(c)/m) < z_0/A_1(c) < (c + 5/c)/A_1(c) \leq 0.0008$. When $c < 5$, since $z_0 < 5.91$ by Lemma 3.2, we have $z_0/A < 0.00016$. In any case, we obtain (11).

Using (11), we have

$$|q_2| = \left| \frac{\pm A + 4z_0}{z_0^2 + 1} \right| < \frac{1.0032A}{z_0^2 + 1}.$$

Also we have

$$q_2\xi_2 + q_1 = -\frac{\pm A + 4z_0}{3z_0} \left(1 + \frac{\varepsilon_2}{A}\right) + 1 = \mp \frac{A}{3z_0} \left(1 \pm \left(1 \pm \frac{\varepsilon_2}{z_0} + \frac{4\varepsilon_2}{A}\right) \frac{z_0}{A}\right),$$

hence, using (11), $z_0 > 7/3$, $|\varepsilon_2| < 1$ and $A \geq 37500$, we have

$$\frac{0.9988A}{3z_0} < |q_2\xi_2 + q_1| < \frac{1.0012A}{3z_0}. \quad (12)$$

Now we estimate from above

$$C_2(c, y) := \frac{m|q_2(q_2\xi_2 + q_1)|}{K(c)by^4}.$$
Using the lower bound for \( y \) given by Lemma 5.1, the above estimates for \( |q_2| \) and (12), Lemma 4.1, \( m \leq 4, \ b \geq 5 \) and \( y_0 \geq 1 \), we obtain

\[ C_2(c, y) < \frac{2.75A_2(c)^2}{z_0(z_0^2 + 1)K(c)^5}. \]

According to the four cases for \( c \), we estimate from above the right-hand side using Lemma 3.2, \( K(c) \) in Lemma 3.3, and \( A_2(c) \) in Lemma 4.1. We can verify that, in each case the upper bound \( c_2 \) for the right-hand side is given by the value at the left boundary of the corresponding interval for \( c \), from which we obtain the lemma.

**Remark 5.2** Actually, if we have an estimate \( c_2 \leq 0.0021 \) for all cases, it is sufficient for later use.

This lemma immediately implies the following.

**Lemma 5.7** Let \((x, y)\) be a solution of (2) with \( y > 0, \ x/y > 1, \) and \( x/y \neq x_0/y_0 \). Suppose \( A \geq 37500 \). Then

\[ y > \frac{0.166A}{z_0}. \]

**Proof.** By Lemma 5.2, \( d_2 = 2 \) and (12), we obtain

\[ y > \frac{|(q_2\xi_2 + g_1)|}{(1 + c_2)d_2} > \frac{0.16646A}{(1 + c_2)z_0}. \]

We see that \( 0.16646/(1 + c_2) > 0.166 \) for the value \( c_2 \) given in Lemma 5.6 for all four cases. Thus the lemma follows.

**Remark 5.3** We can see \( 0.166A/z_0 > \text{constant} \times c_4^4y_0^4/m \). The exponent of \( y_0 \) has increased by 1 from Lemma 5.1. For Case 1, this constant is 0.166.

### 5.5 Comparison with \( x_2/y_2 \)

Next we compare solutions \((x, y)\) of (2) with \( x_2/y_2 \). For it, we prepare the following estimate.

**Lemma 5.8** Let \((x, y)\) be a solution of (2) with \( y > 0, \ x/y > 1, \) and \( x/y \neq x_0/y_0 \). Suppose \( A \geq 37500 \). Then

\[ \frac{m}{K(c)y^4} < \frac{c_3}{|q_2(q_3\xi_3 + q_2)|}, \]

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where \( c_3 \) is given as follows:

- Case 1. \( c_3 = 0.00642 \),
- Case 2. \( c_3 = 0.0198 \),
- Case 3. \( c_3 = 0.1202 \),
- Case 4. \( c_3 = 0.298 \).

**Proof.** Using (11), we have

\[
|q_3| = \left| \frac{\mp A - z_0}{3z_0} \right| < \frac{1.0008A}{3z_0}.
\]

We also have

\[
q_3 \xi_3 + q_2 = \frac{\mp Ay_0 - x_0}{3x_0} \cdot \frac{\pm 9x_0^2 A}{5(x_0^2 + y_0^2)} \left( 1 + \frac{2\varepsilon_3}{A} \right) - \frac{(\pm Ay_0 + 4x_0)y_0}{x_0^2 + y_0^2}.
\]

Using (11), \( z_0 > 7/3 \), \( |\varepsilon_3| < 1 \) and \( A \geq 37500 \), we see that the absolute value of the quantity in the last parentheses is smaller than 0.00088. Hence we have

\[
\frac{0.99912 \times 3z_0 A^2}{5(z_0^2 + 1)} < |q_3 \xi_3 + q_2| < \frac{1.00088 \times 3z_0 A^2}{5(z_0^2 + 1)}. \tag{13}
\]

Now we estimate from above

\[
C_3(c, y) := \frac{m|q_3(q_3 \xi_3 + q_2)|}{K(c)by^4}.
\]

Using the lower bound for \( y \) given by Lemma 5.7, the above estimates for \( |q_3| \) and (13), Lemma 4.1, \( m \leq 4 \), \( b \geq 5 \) and \( y_0 \geq 1 \), we obtain

\[
C_3(c, y) < \frac{168.9z_0^4}{(z_0^2 + 1)K(c)A_1(c)}.
\]

For each of the four cases for \( c \), we estimate from above the right-hand side using Lemma 3.2, \( K(c) \) in Lemma 3.3, and \( A_1(c) \) in Lemma 4.1. We can verify that, in each case the upper bound \( c_3 \) for the right-hand side is given by the value at the left boundary of the corresponding interval for \( c \), from which we obtain the lemma.

This lemma immediately implies the following.
Lemma 5.9 Let \((x, y)\) be a solution of (2) with \(y > 0\), \(x/y > 1\), and \(x/y \neq x_0/y_0\). Suppose \(A \geq 37500\). Then

\[ y > \frac{c_4 A^2}{y_0(z_0^2 + 1)}, \]

where \(c_4\) is given as follows:

Case 1. \(c_4 = 0.198\), Case 2. \(c_4 = 0.195\), Case 3. \(c_4 = 0.178\), Case 4. \(c_4 = 0.153\).

Proof. By Lemma 5.2, \(d_3 = 3x_0\) and (13), we obtain

\[ y > \frac{|(q_3 \xi_3 + q_2)|}{(1 + c_3)d_3} > \frac{0.1998 A^2}{(1 + c_3)y_0(z_0^2 + 1)}. \]

Using \(c_3\) given in Lemma 5.8, we obtain the lemma.

Remark 5.4 We can see \(c_4 A^2/(y_0(z_0^2 + 1)) > \text{constant} \times c_8 b^2 y_0^7/m^2\). For Case 1, this constant is 0.198. The exponent of \(y_0\) has increased to 7. This exponent is bigger than the exponent of \(y_0\) in the upper bound for \(y\) given by Lemma 4.6 (see Remark 4.2). As we will see in the next section, this lower bound is sufficiently large for obtaining a contradiction.

6 Proof of Theorem 2.1

In order to prove Theorem 2.1, we first consider the case \(A \geq 37500\). The case \(A < 37500\) will be treated later.

6.1 Proof of Theorem 2.1 in case \(A \geq 37500\)

We suppose \(A \geq 37500\). We further suppose that there exists a solution \((x, y)\) of (2) such that \(y > 0\), \(x/y > 1\), and \(x/y \neq x_0/y_0\). In this case, we have an upper bound for \(y\) given by Lemma 4.6, and a lower bound for \(y\) given by Lemma 5.9. Combining these, we would have

\[ \frac{c_4 A^2}{y_0(z_0^2 + 1)} < 7.43 B(c)^{1.4} y_0^{1.74} \left( \frac{m A}{K(c)b} \right)^{0.69}, \]

that is,

\[ 1 < \frac{7.43 m^{0.69} B(c)^{1.4} y_0^{2.74} (z_0^2 + 1)}{c_4 b^{0.69} K(c)^{0.69} A^{1.31}} = \frac{7.43 m^{0.69} B(c)^{1.4} y_0^{2.74} (z_0^2 + 1)}{c_4 b^{0.69} K(c)^{0.69} A^{0.685}} \cdot \frac{1}{A^{0.625}}. \]
We replace the first $A$ by Lemma 4.1, and the second $A$ by 37500, then

$$< \frac{7.43m^{0.69}B(c)^{1.4}y_0^{2.74}(z_0^2 + 1)}{c_4b^{0.69}K(c)^{0.69}(by_0^4A_1(c)/m)^{0.685}} \cdot \frac{1}{37500^{0.625}}.$$  

Note that the exponents of $y_0$ in the numerator and the denominator are same. Then by $m \leq 4$ and $b \geq 5$, we would have

$$1 < \frac{0.00757(z_0^2 + 1)B(c)^{1.4}}{c_4K(c)^{0.69}A_1(c)^{0.685}}. \quad (14)$$

We evaluate (14) from above.

For Case 1, by replacing $K(c)$ by $c^3$, and $A_1(c)$ by $c^5$, and by Lemma 3.2 and $c \geq 5$, we see the right-hand side RHS of (14)< 0.033, thus we would obtain a contradiction.

For Case 2, by a similar way of estimate, we see RHS of (14)< 0.296, thus we would obtain a contradiction.

For Case 3, we rewrite RHS of (14) as

$$\frac{0.00757((z_0^2 + 1)/c^2)(B(c)/c^2)^{1.4}/c^{0.695}}{c_4(K(c)/c^3)^{0.69}(A_1(c)/c^5)^{0.685}}.$$ 

For $7/6 \leq c < 3$, we estimate the numerator from above, that is, the value at $c = 7/6$, and estimate the denominator from below, that is, the value at $c = 3$, and we see RHS of (14)< 0.62, thus we would obtain a contradiction.

For Case 4, we estimate the numerator of RHS of (14) from above, that is, the value at $c = 7/6$, and estimate the denominator from below, that is, the value at $c = 0$, and we see RHS of (14)< 0.053, thus we would obtain a contradiction.

Thus for all cases we would obtain a contradiction. Therefore, if $A \geq 37500$, then (2) has no solution $(x, y)$ with $y > 0$ and $x/y > 1$ other than $(x_0, y_0)$. This proves Theorem 2.1 in case $A \geq 37500$.

### 6.2 Proof of Theorem 2.1 in case $A < 37500$

We suppose $A < 37500$. In this case, from Lemma 4.1, we have

$$\frac{1}{m}A_1(c)by_0^4 < A < 37500.$$
We see from this, that for Case 1, \( c \) is bounded from above, hence for all four cases, \( c \) is bounded. We also see that, for all four cases, \( y_0 \) and \( b \) are bounded from above. Since \( c = b/a \), and \( c \) and \( b \) are bounded, we see that \( a \) is also bounded. Since \( c \) is bounded, we see that \( x_0/y_0 \) is bounded by Lemma 3.2, and since \( y_0 \) is bounded, we see that \( x_0 \) is also bounded. Therefore, we have only a finite number of 4-uples \((a, b, x_0, y_0)\) which satisfy \( A < 37500 \). Next for these \((a, b, x_0, y_0)\), we calculate the value of \( F \), and we keep those pairs \((a, b)\) such that this value is \( \pm 1 \) or \( \pm 4 \), and omit the others. Then we have 344 remaining pairs \((a, b)\). Next, using our results by Padé approximation method, we verify that for these finite number of pairs \((a, b)\), (2) has no solution \((x, y)\) with \( y > 0 \) and \( x/y > 1 \) except the above \((x_0, y_0)\), by the following way. From Lemma 4.1, we see \( A \geq 208 \), hence from Lemma 4.4 and Lemma 4.5, we have an upper bound for solutions of (2) with \( y > 0 \) and \( x/y > 1 \). We calculate the value of \( \theta \), its normal continued fraction expansion, and principal convergents up to necessary size given by the upper bound for the solutions, and verify that, except \( x_0, y_0 \), these convergents do not satisfy (2), which concludes that (2) has only one solution in case \( A < 37500 \).

From the above results for both cases of \( A \), the proof of Theorem 2.1 is complete.

Proof of Theorem 1.1 As mentioned in §2, Theorem 2.1 implies Theorem 1.1.

References


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